Part I: Problems

1. A firm’s fixed cost is $260,000; it costs the firm $77 to make each unit of its product; and it sells each unit for $90.
   a) Find its cost, revenue, and profit functions.
   b) How much does each unit made and sold contribute towards profit?
   c) Find its break-even point.
   d) What is happening to the profit at 15,000 units? Why?
   e) What is happening to the profit at 22,000 units? Why?

2. A firm has a product whose total revenue function is given by $R(x) = 175x - \frac{1}{2}x^2$ and total cost function is $C(x) = 3600 + 25x + \frac{1}{2}x^2$. Determine the possible production runs the product can break even.

3. A firm has for a product its total cost given by $C(x) = 0.5x^2 + 10,000$ and the demand function given by $p = 1500 - 2x$. Give its profit function.

4. The cost function for a product is given by $C(x) = x^2 + 200$ and its demand function is $p = 760 - x$, where $x$ is the number of units per month.
   a) Give its revenue and profit functions.
   b) Find the marginal cost, marginal revenue, and marginal profit at 150 units.
   c) Interpret your answers to part b.
   d) Determine the number of units that should be produced and sold per month in order to maximize profit.

5. A product has a demand function given by $p = 480 - 3x$ and supply function given by $p = 17x + 80$.
   a) Find its equilibrium point.
   b) Determine whether a unit price of $400 will yield a surplus or shortage and explain why.

6. A product has a total revenue function given by $R(x) = 500x - 2x^2$ and total cost function given by $C(x) = 3600 + 100x + 2x^2$.
   a) Give the profit function.
   b) Determine the number of units needed to achieve maximum profit.
   c) What is the maximum profit?

7. If $\frac{1}{2} \cdot \log_3 (x - 2) = 2 \cdot \log_3 2 - \frac{1}{2} \cdot \log_3 (x - 2)$, find $x$.

8. Determine the number of years it would take to triple the amount of principle invested at 8.8% compounded continuously.

9. A radioactive substance has a half-life of 26 months. After 10 months, 42 grams remain. Determine the initial amount.
   \[
   A = A_0 \cdot 2^{-\frac{x}{26}}
   \]

10. Determine the future value of an annuity of $40 paid at the end of each 6-month period for 10 years at a 6% rate of interest, compounded semiannually.

11. Determine the future value of an annuity of $200 paid at the beginning of each 6-month period for 8 years at a 6% interest rate, compounded semiannually.

12. A debt of $8,000 is to be amortized with 8 equal semiannual payments at a 12% interest rate, compounded semiannually. Determine the amount of those payments.
13. The Smiths want to have $20,000 in two years for a down payment on a house. To meet this goal, they will make deposits at the end of each month for the next two years into an account that pays 12% compounded monthly.
   a) Find the size of the monthly payments.
   b) What were their total earnings on this account?

14. The Smiths in problem #13 above decide to buy a house priced at $300,000. With the $20,000 accumulated from their sinking fund account and another $10,000 from savings, they make a 10% down payment and take out a loan on the balance at 8% compounded monthly for the next 30 years.
   a) Determine the amount of their monthly payments.
   b) What would be the total interest paid over the 30 years?
   c) After living in their home for 10 years, they found it necessary to relocate due to Mr. Smith’s job. Find the unpaid balance of the loan at this time.

For #15-19, determine the indicated limits:

15. \( \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \) if \( f(x) = 2x^2 - x^3 \)

16. \( \lim_{x \to 2} \frac{x^4 - 16}{4x - 8} \)

17. \( \lim_{h \to 0} \frac{3(a+h)^2 - 3a^2}{h} \)

18. \( \lim_{x \to \infty} \frac{4x^3 - 4x - 3}{3x(2x+1)^2} \)

19. \( \lim_{x \to -2} f(x), \lim_{x \to -2^-} f(x), \lim_{x \to -2^+} f(x) \) if \( f(x) = \begin{cases} 12 - x^2, & x < -2 \\ x^3 + 4x + 8, & x \geq -2 \end{cases} \)

20. If \( f(x) = \begin{cases} x - 2, & x \leq 5 \\ cx - 3, & x > 5 \end{cases} \), determine the value of \( c \) that will allow \( f(x) \) to be continuous.

21. Given: \( f(x) = x^4 - 3x^3 + 3x^2 + 1 \)
   a) Determine the interval(s) where \( f \) is increasing and where it is decreasing.
   b) Determine the interval(s) where \( f \) is concave up and where it is concave down.
   c) Give local extremas and point(s) of inflection, if any exist.
   d) Using the above information, sketch its graph.
   e) Determine the area of the region bounded by the curve and the x-axis and between \( x = -1 \) and \( x = 2 \).

22. If \( f(x) = 5x^2 + 3x \), use the limit definition of the derivative to find \( f'(x) \).

Find the indicated derivatives:

23. \( f'(x) \), if \( f(x) = (x^2 - x)^3 \)

24. \( \frac{dy}{dx} \), if \( y = \left(\frac{3x - 1}{x^2 + 2}\right)^7 \)

25. \( \frac{dy}{dx} \), if \( y = e^{3\ln x} \)

26. \( \frac{dy}{dx} \), if \( e^y = x \)
27. If \( f(x) = \frac{(\ln e)^2}{2} \), find \( f'(e) \).

28. A ball is thrown vertically upward from the ground with an initial velocity of 128 ft/sec. If the positive direction of the distance from the starting point is up, the equation of the motion is \( s(t) = -16t^2 + 128t \).
   a) Find the instantaneous velocity and acceleration of the ball after 3 seconds.
   b) Determine how long it will take the ball to reach its highest point.
   c) What is the maximum altitude attained by the ball?
   d) What is the maximum velocity attained by the ball?

Determine the equation of the tangent line that satisfies the following conditions.

29. Tangent to the graph of \( x^2 + y^2 = 26y \) at \((5, 1)\).

30. Tangent to the curve \( y = 4x^2 - x \) that has slope of 7.

31. An efficiency study was conducted for the postal system of a large metropolitan city. It was found that the number of pieces of mail sorted by the average worker \( t \) hours after starting work at 8:00 a.m. is given by \( N(t) = -2t^3 + 15t^2 + 20 \), where \( 0 \leq t \leq 4 \). At what time during the morning shift is the average worker performing at peak efficiency?

32. Using the curve-sketching procedure discussed in class, sketch the graph of \( y = \frac{(x-1)^2}{x^2} \).
   (Domain, intercepts, asymptotes, sign analysis to determine local extremas, concavity)

33. The total area of a circle with radius \( x \) and a square with side \( 10 - x \) is given by the formula \( A = \pi x^2 + (10 - x)^2 \). Determine the value of \( x \) that will minimize the total area.

34. A rectangular box with a square base and a volume of 20 \( \text{ft}^3 \) is required to ship an item. The material for the base costs 40 cents/\( \text{ft}^2 \), the sides costs 15 cents/\( \text{ft}^2 \), and the top costs 35 cents/\( \text{ft}^2 \).
   a) If \( x \) represents the length of a side of the square base, express the surface area \( S \) of the box as a function of \( x \).
   b) Express the total cost \( C \) of the required material as a function of \( x \).
   c) Give the domain of \( C \).
   d) Determine the dimensions of the box that can be constructed for the least cost.
   e) Find the cost of the box if the above dimensions are to be used.

35. If the profit of a product is presented by \( P(x) = -0.01x^2 + 178x - 100 \) and if the profit is increasing at the rate of \$4000 per month, determine the approximate rate of change of sales with respect to time (in units/month) when the monthly sales are \( x = 300 \) units.
36. The monthly demand function for a product is given by \( p = 12x - \frac{x^2}{500} \). The manufacturer is increasing the production of this product at the rate of 50 units per month.
   a) Give the revenue function.
   b) Find the rate of change of the revenue with respect to time (in months) when the monthly production is 1250 units.

37. Use differentials to approximate \( 4\sqrt{14.8} \) (to four decimal places).

38. The cost function for the production of a product is given by \( C(x) = \frac{15,000x}{75 - x} \). Use the differential of \( C \) to determine the approximate change in cost as production is increased from 100 to 102 units.

39. If \( y = \frac{e^{2x}(x-3)^2}{(x+5)^4} \), use logarithmic differentiation to find \( \frac{dy}{dx} \).

40. Consider the demand equation \( x = 3125 - 5p^2 \), where \( 0 \leq p \leq 25 \).
   a) Determine when the demand will be unitary, the interval where demand will be inelastic, and the interval where it will be elastic.
   b) In each of the above three cases, tell what happens to the total revenue when the price decreases.

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Evaluate:

41. \( \int (2 + \sqrt{x})^2 \, dx \)

42. \( \int (3x^3 + 4)^3 \cdot 3x^2 \, dx \)

43. \( \int x \sqrt{x - 4} \, dx \)

44. \( \int x e^{2x^2} \, dx \)

45. \( \int e^x \frac{1}{x(ln(x))^2} \, dx \)

46. \( \int_0^1 x e^{2x} \, dx \)

47. If \( G''(x) = \frac{1}{e^x} + 2 \), with \( G'(0) = 3 \), and \( G(0) = 2 \), find \( G(x) \).

48. If \( f(x) = 2 - x^2 \), compute the Riemann sum of \( f \) over the interval \([2,4]\) using four subintervals of equal length. Choose the representative points to be the midpoints of the subintervals.

49. The marginal revenue for a product is given by \( R'(x) = \frac{-30}{(2x+1)^2} + 30 \). Determine its revenue function \( R(x) \).

50. The marginal cost function of a product is given by \( \frac{dC}{dx} = \frac{89x^2}{x^3 + 1} \), where \( x \) is the number of units. Determine the approximate total cost of the first 400 units.

51. Find the area of the region bounded by \( f(x) = \frac{1}{x} \), the x-axis, \( x = e \), and \( x = 5e \).
52. Find the area of the region bounded by \( f(x) = \frac{1}{2}x + 3 \) and \( f(x) = x^2 \).

53. A product has a supply function given by \( p = x + 1 \) and a demand function given by \( p = \sqrt{49 - 6x} \). Find a) its consumers’ surplus and b) its producers’ surplus if production is set at market equilibrium.

54. Suppose gold is being produced at a rate of \( y = \frac{25}{t + 0.5} + 3 \) metric tons per year \( t \), where \( t = 0 \) represents the present. Find the area under this function from \( t = 1 \) to \( t = 4 \) and interpret your result.

(Answers to Part I)

1. a) \( C(x) = 77x + 260,000 \); \( R(x) = 90x \); \( P(x) = 13x - 260,000 \)  b) $13  c) (20,000, 1,800,000)
   d) Since 15,000<20,000, there will be a loss. \([ \text{Also, } P(15,000) = -65,000] \)
   e) Since 22,000>20,000, there will be a profit. \([ \text{Also, } P(22,000) = 26,000] \)

2. 30 units; 120 units

3. \( P(x) = -2.5x^2 + 1500x - 10,000 \)

4. a) \( R(x) = 760x - x^2 \); \( P(x) = -2x^2 + 760x - 200 \)  b) \( C'(150) = 300; \ R'(150) = 460; \ P'(150) = 160 \)
   c) increase in cost of $300 on production of 151st unit; increase in revenue on sale of 151st unit; increase in profit of $160 on sale of 151st unit
   d) 190 units

5. a) (20,420)  b) shortage; demand > supply

6. 50 units

7. 4

8. 12.5 years

9. 255 grams

10. $1074.81

11. $4152.32

12. $1288.29

13. a) Sinking fund; $741.46  b) $2,204.96

14. a) $1981.16  b) $443,217.60  c) $236,856.19

15. \( 4x - 3x^2 \)

16. 8

17. 6a

18. 0

19. 8, -8, DNE

20. \( \frac{6}{5} \)

21. a) increasing on \((0, \infty)\); decreasing on \((-\infty, 0)\)
   b) concave upward on \((-\infty, \frac{1}{2})\) and \((1, \infty)\);
   concave downward on \(\left(\frac{1}{2}, 1\right)\)
   c) no local maximum
   local minimum at \((0,1)\)
   points of inflection at \(\left(\frac{1}{2}, \frac{27}{16}\right)\) and \((1,2)\)
   d) \(\frac{147}{20} \approx 7.35 \text{ sq. units}\)

22. \( 10x + 3 \)

23. \( 3(x^2 - x)^2 (2x - 1) \)

24. \( \frac{-7(x^2 - 1)^6 (3x^2 - 2x - 6)}{(x^2 + 2)^8} \)

25. \( 3x^2 \)

26. \( \frac{1}{x} \)

27. \( \frac{2}{e} \)
28. a) $v(3) = 32 \text{ ft/sec}; \ a(3) = -32 \text{ ft/sec}^2$  
   b) 4 seconds  
   c) $s(4) = 256 \text{ ft}$  
   d) 128 ft/sec

29. $y = \frac{5}{12} x - \frac{13}{12}$  
30. $y = 7x - 4$  
31. 10:30 a.m.  
33. $\frac{10}{\pi + 1}$

32. Vertical asymptote at $x = 0$  
   Horizontal asymptote at $y = 1$  
   Crossover at $(\frac{1}{7}, 1)$  
   Critical value at $x = 1$  
   Local minimum at $(1, 0)$

34. a) $S(x) = \frac{80}{x} + 2x^2$  
   b) $C(x) = \frac{12}{x} + 0.75x^2$  
   c) $x > 0$  
   d) 2' x 2' x 5'  
   e) $9.00$

35. increasing 23 units/month  
36. a) $R(x) = 12x^2 - \frac{x^3}{500}$  
   b) $\$1,031,250$

37. 1.9614  
38. $\$3600$  
39. $\frac{dy}{dx} = \frac{2e^{2x}(x-3)(x^2+x-4)}{(x+5)^3}$

40. a) unitary when $p = $14.43; inelastic when $0 \leq p < $14.43; elastic when $14.43 < p \leq $25  
   b) unitary: revenue does not change; inelastic: revenue decreases; elastic: revenue increases

41. $4x + \frac{8}{3}x^{3/2} + \frac{1}{2}x^2 + C$  
42. $-\frac{1}{6}(3x^3 + 4)^2 + C$  
43. $\frac{2}{5}(x-4)^{3/2} + \frac{8}{3}(x-4)^{1/2} + C$

44. $\frac{1}{4}e^{2x^2} + C$  
45. $\frac{2}{3}$  
46. $\frac{1}{4}(e^x + 1)$  
47. $G(x) = e^{-x} + x^2 + 4x + 1$

48. -14.625  
49. $R(x) = \frac{15}{(2x+1)} + 30x - 15$  
50. $\$533.24$

51. ln 5 $\approx 1.609$ sq. units  
52. $\int_{1/2}^{2} \left[ (\frac{1}{2}x + 3) - x^2 \right] dx = \frac{49}{16} \approx 3.06$ sq. units

53. a) $\$4.22$  
   b) $\$8.00$  
54. 36.5 metric tons; total production from $t = 1$ to $t = 4$ years
Part II: Multiple Choice

1. The graph of \( f(x) = x^3 + 27x \) has
   (A) One local maximum, one local minimum, one point of inflection.
   (B) One local minimum, no local maximum, one point of inflection.
   (C) One local minimum, no local maximum, one point of inflection
   (D) One point of inflection only
   (E) None of these

2. The differential of \( p \) for the demand function \( p(x) = \sqrt[3]{80 - 7x} \) is
   (A) \( \frac{7dx}{2\sqrt[3]{80 - 7x}} \)  
   (B) \( \frac{-dx}{2\sqrt[3]{80 - 7x}} \)  
   (C) \( \frac{-7dx}{2\sqrt[3]{80 - 7x}} \)  
   (D) \( \frac{dx}{2\sqrt[3]{80 - 7x}} \)  
   (E) none of these

3. If, for all values of \( x \), \( f'(x) < 0 \) and \( f''(x) > 0 \), which of the following curves could be part of the graph of \( f \)?
   (A) ![Graph A]  
   (B) ![Graph B]  
   (C) ![Graph C]  
   (D) ![Graph D]  
   (E) ![Graph E]

4. For the function \( f \) defined, at which point is \( f''(x) \) not positive?
   (A) A  
   (B) B  
   (C) D  
   (D) F  
   (E) none of these

5. If a commodity’s elasticity of demand at price \( p \) is given by \( E(p) \) and \( E(p) = 1.35 \), then which of the following is true?
   (A) Increasing the price by a small amount will result in a decrease in revenue.
   (B) Increasing the price by a small amount will result in an increase in revenue.
   (C) Increasing or decreasing the price by a small amount will leave revenue approximately constant.
   (D) Cannot be determined.
   (E) None of these

(Answers to Part II)

1. D  
2. C  
3. E  
4. B  
5. A