Dr. Nestler - Math 2 - 3.2 - Real Zeros of Polynomials

**Theorem.** If \( f(x) \) is a polynomial and \( c \) is a real number, then the following statements are equivalent, meaning that either they are all true or they are all false:

1. \( c \) is a zero of \( f \), meaning \( f(c) = 0 \).
2. \( x - c \) is a factor of \( f(x) \), meaning the remainder of \( f(x) \) divided by \( x - c \) is zero.
3. \( c \) is an \( x \)-intercept of the graph of \( f(x) \), meaning the point \((c, 0)\) is on the graph of \( f \).

**Example:** \( f(x) = x^2 + x - 6 = (x + 3)(x - 2) \).

- \( x + 3 \) is a factor of \( f(x) \) \( \Leftrightarrow \) \( f(-3) = 0 \) \( \Leftrightarrow \) \(-3\) is an \( x \)-intercept of the graph of \( f(x) \).
- \( x - 2 \) is a factor of \( f(x) \) \( \Leftrightarrow \) \( f(2) = 0 \) \( \Leftrightarrow \) \( 2 \) is an \( x \)-intercept of the graph of \( f(x) \).

Since a polynomial of degree \( n \) has at most \( n \) linear factors, it has at most \( n \) real zeros.

This means that the number of real zeros must be equal to or less than its degree.

**Examples:** \( f(x) = \frac{7}{8}x(x - 1.23)(2x + 5) \)

\[ g(x) = -4(x - \frac{4}{5})^{13} \]

\[ h(x) = x^2 + 9 \]
**Example:** Find the quotient and remainder of

\[ f(x) = 3x^5 + 5x^4 - 4x^3 + 7x + 3 \text{ divided by } x + 2. \]

Synthetic division:

**Example:** Is the number 1 a zero of \( f(x) = 2x^3 - x^2 + 2x - 3? \)

1 is a zero of \( f(x) \iff x - 1 \) is a factor of \( f(x) \iff \text{remainder of } \frac{f(x)}{x-1} \) is zero

**Rational Zeros Theorem.** Suppose \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \)

is a polynomial of degree \( n \geq 1 \) where the coefficients are all integers and the constant term \( a_0 \) is not zero. If \( \frac{p}{q} \) is a reduced rational zero of \( f \), then \( p \) is a factor of the constant term \( a_0 \), and \( q \) is a factor of the leading coefficient \( a_n \).

**Idea of Proof:** Outlined in 3.2 #118. A complete proof requires what is called mathematical induction, a topic in section 11.4 that we will study at the end of the course.