$SK_1$ of Affine Curves over Finite Fields

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We show that $SK_1(X) = 0$ for every affine curve $X$ over a finite field.

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1. INTRODUCTION

Let $k$ be a field and $A$ a commutative, finitely generated $k$-algebra of Krull dimension one. Let $X = \text{Spec } A$ and call $X$ an affine curve over $k$.

We briefly recall the definitions of $K_1$ and $SK_1$ of a commutative ring $R$. For each positive integer $n$, let $E_n(R)$ be the subgroup generated by elementary matrices (those with ones on the diagonal and at most one non-zero element elsewhere) in the general linear group $GL_n(R)$. There are natural inclusions $GL_n(R) \rightarrow GL_{n+1}(R)$ and $E_n(R) \rightarrow E_{n+1}(R)$. Let $GL(R)$ and $E(R)$ denote the direct limits of the $GL_n$ and $E_n$, respectively. The group $K_1(R)$ is defined to be $GL(R)/E(R)$, which is the maximal abelian quotient of $GL(R)$, and $SK_1(R)$ is the kernel of the surjective determinant map $K_1(R) \rightarrow R^*$. If $X = \text{Spec } R$ is an affine curve, then $SK_1(X) = SK_1(R)$.

In the case $k = R$, $SK_1(X)$ can be very large; for example, if $X$ is a node then $SK_1(X)$ is a real vector space of uncountably infinite dimension ([9, Prop. 121]). When $k$ is a number field, it is conjectured that $SK_1(X)$ is torsion ([3, Remark 1.24], [14, p. 137]). Throughout this paper, $k$ is a finite field. In this case it is known that $SK_1(X)$ is finite ([8, Kor. 3.23]), and trivial when $X$ is smooth ([2, Cor. 4.3]) or a simple cusp ([9, p. 33]) or a union of affine lines ([7, Thm. 211], [13, Thm. 1]). Krusmeyer asked whether $SK_1(X) = 0$ always ([9, p. 32], [10, p. 80]). In this paper we prove this using excision.
For an affine curve $X$, $SK_1(X) = SK_1(X_{\text{red}})$, so we may assume below that our curves are reduced ([1, Cor. 9.2, p. 267]). Moreover, since $SK_1$ commutes with products, we may assume that our curves are connected.

If $A$ is the coordinate ring of a reduced curve over a perfect field of
positive characteristic and $B$ is the normalization of $A$ (i.e. the integral
closure of $A$ in its total quotient ring), and $I$ is an ideal of $B$ contained
in $A$, then excision holds ([6, Thm. 4.2]) and so there is a Mayer-Vietoris
exact sequence as follows:

$$K_2(B/I) \rightarrow K_1(A) \rightarrow K_1(B) \oplus K_1(A/I) \rightarrow K_1(B/I).$$

In fact further diagram chasing reveals that $K_1$ may be replaced with
$SK_1$, giving the following exact sequence:

$$K_2(B/I) \rightarrow SK_1(A) \rightarrow SK_1(B) \oplus SK_1(A/I) \rightarrow SK_1(B/I). \quad (1)$$

This sequence is the main tool used in the proofs below.

2. RESULTS

**Lemma 2.1.** Let $X$ be an irreducible affine curve over a finite field. Then
$SK_1(X) = 0$.

**Proof.** Let $B$ be the normalization of the integral domain $A$, and let
$I$ be the conductor of the extension $B/A$. Since $B$ is a finitely generated
$A$-module, $I$ is not zero ([5, Cor. 13.13]). Write $I = \prod_{i=1}^r P_i^{a_i}$ for prime
ideals $P_i$ of the Dedekind domain $B$, and $a_i \geq 1$ for each $i$.

As mentioned above, since excision holds, the cartesian square

$$
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
A/I & \rightarrow & B/I
\end{array}
$$

induces the Mayer-Vietoris exact sequence (1). Since $SK_1(B) = 0$, it
suffices to show that $SK_1(A/I) = 0$ and $K_2(B/I) = 0$.

If $S/R$ is an integral ring extension and $J$ is an ideal of $S$, then $S/J$ is
integral over $R/(J \cap R)$. Since the conductor $I$ is an ideal of both $A$ and
$B$, $B/I$ is integral over $A/I$. This gives $\dim (A/I) = \dim (B/I) = 0$, as
the quotient ring $B/I$ is Artinian (here we use $I \neq 0$). The quotient ring
$A/I$ is Noetherian and 0-dimensional, so Artinian. Thus $A/I$ is semilocal,
so $SK_1(A/I) = 0$ ([1, Cor. 9.2, p. 267]).

Finally we show that $K_2(B/I) = 0$. The field $L_i = B/P_i$ is a quotient
of $k[x_1, \ldots, x_n]$, that is, a finitely generated $k$-algebra. Hence $L_i$
is a finite algebraic extension of $k$, and thus a finite field. Thus $L_i$ is separable (in fact Galois) over $k$. Thus $B/P^{m_i}_i \simeq L_i[u]/u^m$ for positive integers $m$ ([6, Lemma 2.2]), so in particular $B/P^{m_i}_i \simeq L_i[u]/u^{a_i}$. Now $B/I = B/P^{m_i}_i \simeq \prod_{i=1}^{r} B/P^{m_i}_i$, so $K_2(B/I) = 0$ ([4, Cor. 4.4(a)]).

**Proposition 2.1.** Let $X$ be an affine curve over a finite field. Then $SK_1(X) = 0$.

**Proof.** We may assume that $X = \text{Spec } A$ is reducible. Let $B$ be the normalization of $A$ as before. Then $B$ is a product $\prod_{i=1}^{r} B_i$ of Dedekind domains, with $r \geq 2$. The conductor $I$ of $B/A$ is an ideal of $B$, so $I$ is a direct product $\prod_{i=1}^{r} I_i$ of ideals of the rings $B_i$.

The cartesian square

\[
\begin{array}{ccc}
A & \rightarrow & B = \prod_{i=1}^{r} B_i \\
\downarrow & & \downarrow \\
A/I & \rightarrow & B/I \simeq \prod_{i=1}^{r} B_i/I_i
\end{array}
\]

and excision give rise to a Mayer-Vietoris exact sequence (1) as before.

We again will show that the outer groups are trivial. First, Spec $B$ is a smooth curve over a finite field so $SK_1(B) = 0$.

Next, we show that $SK_1(A/I) = 0$, using the inequality

$$\text{ht}(I) + \dim(A/I) \leq \dim(A) = 1$$

where $\text{ht}(I)$ is the height of the ideal $I$. If $\text{ht}(I) = 0$, then $I$ would be contained in a minimal prime ideal of $A$. The Noetherian ring $A$ has only finitely many minimal prime ideals, and they contain only 0 and the zero divisors of $A$ ([11, Prop. 4.10, p. 26]). However, the conductor $I$ contains a denominator of the total quotient ring of $A$, which is nonzero and not a zero divisor. Hence we must have $\text{ht}(I) = 1$ and so $\dim(A/I) = 0$, giving $SK_1(A/I) = 0$.

For each $i$, $I_i$ is the image of $I$ in $B_i$, so the surjection $A \rightarrow B_i$ sending $I$ to $I_i$ induces a surjection $A/I \rightarrow B_i/I_i$. Thus $\dim(B_i/I_i) \leq \dim(A/I) = 0$, so $\dim(B_i/I_i) = 0$ and thus $I_i \neq 0$. Each $B_i/I_i$ is then Artinian, and thus so is $B/I$. Finally, $K_2(B/I) = \prod_{i=1}^{r} K_2(B_i/I_i)$, and each $K_2(B_i/I_i) = 0$, as in the proof of the Lemma.

**Corollary 2.1.** If $n \geq 3$ and $A$ is a commutative 1-dimensional algebra over a finite field, then any matrix $M \in SL_n(A)$ is necessarily a product of elementary matrices.
Proof. This result follows from the Proposition and the stability results of Bass-Milnor-Serre ([2, Cor. 11.3].

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REFERENCES