Section 3.1: Basic Rules of Differentiation

1. \( f(x) = 2x^2 - x - 3 \) has a horizontal tangent when:
   a. \( x = -1 \)  
   b. \( x = 3/2 \)  
   c. \( x = -1/4 \)  
   d. \( x = 1/4 \)  
   e. None of these

2. Find the equation of the tangent line to \( f(x) = -\frac{5}{2}x^2 + 2x + 2 \) at the point (-1, -5/2)

3. The altitude (in feet) of a rocket t seconds into flight is given by \( f(t) = -t^3 + 6t^2 + 15t \), \( t \geq 0 \)
   a. Find an expression for the rocket’s velocity at anytime \( t \).
   b. What is the average velocity between \( t = 2 \) and \( t = 3 \).
   c. When is the velocity zero (what does this mean with respect to the height of the rocket?)

Section 3.2: Product and Quotient Rules

1. What is the derivative of \( f(x) = \frac{\sqrt{x}}{x^2 + 1} \)
   a. \( f'(x) = \frac{1 - 3x^2}{2\sqrt{x}(x^2 + 1)^2} \)
   b. \( f''(x) = \frac{1}{2}x^{-\frac{1}{2}} \)
   c. \( f'(x) = \frac{-3x^2}{(x^2 + 1)^2} \)
   d. \( f'(x) = \frac{x^{1/2}}{(x^2 + 1)^2} \)
   e. None of these.

2. The concentration of a certain drug in a patient’s bloodstream \( t \) hours after injection is given by \( C(t) = \frac{0.2t}{t^2 + 1} \) milligrams per cubic centimeter. How fast is the concentration changing \( 1/2 \) hour after the injection?
   a. .096 mg/cm³
   b. .08 mg/cm³
   c. .5 mg/cm³
   d. .2 mg/cm³
   e. None of these.

3. \( \lim_{h \to 0} \frac{2(x + h)((x + h)^2 + 1) - 2x(x^2 + 1)}{h} = \)
Section 3.3: Chain Rule

1. Find the derivative for the following (be able to simplify): \( f(x) = \frac{x+1}{\sqrt{3x+2}} \)

2. Find the derivative for the following (be able to simplify) \( f(x) = (3x^2 - 4x)^4(3x-1)^2 \)

3. The pulse rate (the number of heartbeats per minute) of a long-distance runner \( t \) seconds after leaving the starting line is given by \( P(t) = \frac{300 \sqrt{\frac{1}{2} t^2 + 2t + 25}}{t+25}, \ t \geq 0 \)
   Compute \( P'(t) \). How fast is the athlete’s pulse rate increasing 10 seconds into the run? 60 seconds into the run? 2 minutes into the run? What is her pulse rate 2 minutes into the run?

Section 3.4: Marginal Functions in Economics

1. The quantity of Sicard wristwatches demanded per month is related to the unit price by the equation. \( p = \frac{50}{0.01x^2 + 1}, \ 0 \leq x \leq 20 \) where \( p \) is measured in dollars and \( x \) in units of a thousand.
   a. Find the revenue function.
   b. Find the marginal revenue function
   c. Find the marginal revenue at \( x = 2 \) and interpret your result.

2. A small machine shop manufactures drill bits in the petroleum industry. The shop manager estimates that the total daily cost (in dollars) of producing \( x \) bits is \( C(x) = 1000 + 25x - 0.1x^2 \), find the following:
   a. The average cost function.
   b. The marginal average cost function.
   c. Find \( C(10) \) and \( C'(10) \) and interpret your results.
   d. Use the results of part c to estimate the average cost per bit at a production level of 11 bits per day

3. The weekly demand for the Lectro-Copy photocopying machine is given by the demand equation \( p = 2000 - 0.04x \) \( (0 \leq x \leq 50,000) \) where \( p \) denotes the wholesale unit price in dollars and \( x \) denotes the quantity demanded. The weekly total cost function for manufacturing these copiers is given by \( C(x) = .000002x^3 - .02x^2 + 1000x + 120,000 \) where \( C(x) \) denotes the total cost incurred in producing \( x \) units.
   a. Find the revenue function \( R \), the profit function \( P \), and the average cost function \( \overline{C} \).
   b. Find the marginal cost function \( C' \), the marginal revenue \( R' \), the marginal profit \( P' \) and the marginal average cost function \( \overline{C}' \).
   c. Compute \( C'(3000) \), \( R'(3000) \), and \( P'(3000) \) and interpret your results.
4. The proprietor of the Showplace, a video club, has estimated that the rental price $p$ (in dollars) of prerecorded video cassette tapes is related to the quantity $x$ rented per week by the demand equation 

$$x = \frac{1}{4}\sqrt{49 - p^2} \quad \text{where} \quad (0 \leq p \leq 7)$$

a. Find the elasticity when $p = $2.
b. Is the demand elastic or inelastic at this rental price?
c. If the rental price is increased, will the revenue increase or decrease?
d. Determine the interval where the demand will be unitary, the interval where demand will be inelastic, and the interval where it will be elastic.
e. In all of the above three cases, tell what happens the total revenue when the price increases.

Section 3.5: Higher-Order Derivatives

1. Find the first and second derivatives for $f(x) = x(x^2 + 1)^2$

2. Find the first and second derivatives for $f(x) = \frac{x^2}{x - 1}$

Section 3.6: Implicit Differentiation and Related Rates

1. Find the equation of the line tangent to $x^2y^3 - y^2 + xy - 1 = 0$ at $(1,1)$
   a. $y = -(3/2)x + 5/2$  
   b. $y = x + 1$  
   c. $y = 1$  
   d. $y = -3/2$  
   e. None of these.

2. Suppose that the wholesale price of a certain brand of medium-size eggs $p$ (in dollars per carton) is related to the weekly supply $x$ (in thousands of cartons) by the equation $625p^2 - x^2 = 100$. If 25,000 cartons of eggs are available at the beginning of a certain week and the price is falling at the rate of 2 cents per carton per week, at what rate is the supply falling?

3. Find $dy/dx$ using implicit differentiation $\frac{x + y}{x - y} = 3x$

4. Find $dy/dx$ using implicit differentiation $\left(2x + 3y\right)^{\frac{1}{3}} = x^2$

5. A 13-foot ladder is leaning against a wall. If the bottom of the ladder is pulled away from the wall at a rate of 2.5 ft/sec, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 12 feet from the wall.

Section 3.7: Differentials

1. Let $f$ be a function defined by $y = f(x) = \sqrt{2x + 1}$
   a. Find the differential of $f$.
   b. Use your result from (a) to find the approximate change in $y$ if $x$ changes from 4 to 4.1
   c. Find the actual change in $y$ of $x$ changes from 4 to 4.1 and compare your result with that obtained in (b).

2. Use differentials to approximate $\sqrt{26.1}$ to two decimal places.
### Answers Section 3.1: Basic Rules of Differentiation

1. \( f(x) = 2x^2 - x - 3 \) has a horizontal tangent when:

   
   \[ 2x^2 - x - 3 = 0 \]

   \[ x = \frac{1}{4} \quad \text{and} \quad x = \frac{3}{2} \]

   \[ f'(x) = 4x - 1 \]

   \[ f'(x) = 0 \quad \text{when} \quad x = \frac{1}{4} \]

   \[ x = \frac{1}{4} \]

   \[ f''(x) = 4 \quad \text{which is positive, so the tangent line is horizontal at} \quad x = \frac{1}{4} \]

   \[ f'(x) = 0 \quad \text{when} \quad x = \frac{3}{2} \]

   \[ x = \frac{3}{2} \]

   \[ f''(x) = 4 \quad \text{which is positive, so the tangent line is horizontal at} \quad x = \frac{3}{2} \]

   

d. \( x = \frac{1}{4} \), find \( f'(x) \) and set equal to 0 and solve.

2. \( f'(x) = -5x + 2 \), so the slope of tangent is \( f'(-1) = -5(-1) + 2 = 7 \)

   \[ f''(x) = -5 \quad \text{which is negative, so the tangent line is horizontal at} \quad x = -1 \]

   \[ f''(x) = -5 \quad \text{which is negative, so the tangent line is horizontal at} \quad x = 1 \]

   

So the tangent line is

\[ y - \frac{5}{2} = 7(x - (1)) = 7x + \frac{5}{2} = 7x + \frac{9}{2} \]

3. a. Find and expression for the rocket’s velocity at anytime \( t \). \( f'(t) = -3t^2 + 12t + 15 \)

   
   b. What is the average velocity between \( t = 2 \) and \( t = 3 \)

   \[ \frac{f(3) - f(2)}{3 - 2} = \frac{72 - 46}{1} = 26 \text{ ft/sec.} \]

   c. When is the velocity zero (what does this mean with respect to the height of the rocket?) Solve \( f'(t) = -3t^2 + 12t + 15 = 0 \) which occurs when \( t = -1 \) and \( t = 5 \).

   Because time is not negative, \( t = 5 \) is our answer. When the velocity is zero, the rocket is at its maximum height, the rocket changes directions at \( t = 5 \). The maximum height is found by finding \( f(5) = 100 \).

### Answers Section 3.2: Product and Quotient Rules

1. What is the derivative of \( f(x) = \frac{\sqrt{x}}{x^2 + 1} \)

   \[ a. \quad f'(x) = \frac{1 - 3x^2}{2\sqrt{x}(x^2 + 1)^2} \]

2. The concentration of a certain drug in a patient’s bloodstream \( t \) hours after injection is given by \( C(t) = \frac{0.2t}{t^2 + 1} \) milligrams per cubic centimeter. How fast is the concentration changing \( \frac{1}{2} \) hour after the injection? \( a. \quad 0.096 \text{ mg/cm}^3 \)

   Find \( C'(1/2) \)

3. \[ \lim_{h \to 0} \frac{2(x + h)((x + h)^2 + 1) - 2x(x^2 + 1)}{h} = f'(x) \]

   So \( f'(x) = 2x^2 + 1 + 2x(2x) = 2x^2 + 2 + 4x^2 = 6x^2 + 2 \)

### Answers Section 3.3: Chain Rule

1. \[ f''(x) = \frac{1}{2} \left( \frac{x + 1}{3x + 2} \right)^{\frac{1}{2}} \left( 1(3x + 2) - (x + 1)(3) \right) = \frac{-1}{2\sqrt{x + 1}(3x + 2)^{\frac{3}{2}}} \]

2. \[ f'(x) = 4(3x^2 - 4x)^3(6x - 4)(3x - 1)^2 + (3x^2 - 4x)^4(2)(3x - 1)(3) = 2(3x^2 - 4x)^3(3x - 1)(45x^2 - 48x + 8) \]

3. Find
\[
P'(t) = \frac{300 \left( \frac{1}{2} t^2 + 2t + 25 \right)^{\frac{1}{2}}}{(t + 25)^2} (t + 2) (t + 25) - 300 \sqrt{\frac{1}{2} t^2 + 2t + 25} \\
= \left( \frac{1}{2} t^2 + 2t + 25 \right)^{\frac{1}{2}} \left[ 150(t + 2)(t + 25) - 300 \left( \frac{1}{2} t^2 + 2t + 25 \right) \right] \div (t + 25)^2 \\
= \frac{150t^2 + 3750t + 7500 - 150t^2 - 600t - 7500}{(t + 25)^2} = \frac{3450t}{(t + 25)^2}
\]
and then find
\[
P'(10) = 2.889 = 2.9 \text{ beat/min}^2 \\
P'(60) = .64963 \text{ beat/min}^2 \\
P'(120) = .2279 \text{ beats/min}^2 \\
P(120) = 179 \text{ beats/min.}
\]

### Answers Section 3.4: Marginal Functions in Economics

1. a. Find the revenue function. \( R(x) = xp = \frac{50x}{0.01x^2 + 1} \)

b. Find the marginal revenue function
   \[ R'(x) = \frac{50(.01x^2 + 1) - 50x(.02x)}{(0.01x^2 + 1)^2} = \frac{50 - .5x^2}{(0.01x^2 + 1)^2} \]

c. Find the marginal revenue at \( x = 2 \) and interpret your result.
   \[ R'(2) = \frac{50 - .5(2)^2}{(0.01(2)^2 + 1)^2} = 44.39 \]
   The approximate additional revenue from the sale of the 2001st wristwatch is $44.39.

2. a. \( \bar{C}(x) = \frac{1000 + 25x - 0.1x^2}{x} = \frac{1000}{x} + 25 - 0.1x \)

b. \( \bar{C}'(x) = -\frac{1000}{x^2} - 0.1 \)

c. \( \bar{C}(10) = \frac{1000 + 25(10) - 0.1(10)^2}{(10)} = 124 \) or $124

At a production level of 10 bits per day, the average cost of producing a bit is $124.

\( \bar{C}'(10) = -\frac{1000}{(10)^2} - 0.1 = -10.1 \) or -$10.10   At a production level of 10 bits per day, the average cost is decreasing at a rate of $10.10 per bit.

d. If the production level is increased by one bit, then the average cost per bit will decrease by approximately $10.10 dollars, so the average cost per bit at a production level of 11 bits per day is approximately $124 - $10.10 = $113.90.
3. The weekly demand for the Lectro-Copy photocopying machine is given by the demand equation $p = 2000 - 0.04x$ ($0 \leq x \leq 50,000$) where $p$ denotes the wholesale unit price in dollars and $x$ denotes the quantity demanded. The weekly total cost function for manufacturing these copiers is given by $C(x) = 0.000002x^3 - 0.02x^2 + 1000x + 120,000$ where $C(x)$ denotes the total cost incurred in producing $x$ units.

a. Find the revenue function $R$, the profit function $P$, and the average cost function $\overline{C}$.

$$R(x) = x(2000 - 0.04x) = 2000x - 0.04x^2$$

$$P(x) = R(x) - C(x) = 2000x - 0.04x^2 - (0.000002x^3 - 0.02x^2 + 1000x + 120,000) =$$

$$-0.000002x^3 - 0.02x^2 + 1000x - 120,000$$

$$\overline{C}(x) = \frac{0.000002x^3 - 0.02x^2 + 1000x + 120,000}{x} = 0.000002x^2 - 0.02x + 1000 + 120,000x^{-1}$$

b. Find the marginal cost function $C'$, the marginal revenue $R'$, the marginal profit $P'$ and the marginal average cost function $\overline{C}'$.

$$C'(x) = 0.000006x^2 - 0.04x + 1000$$

$$R'(x) = 2000 - 0.08x$$

$$P'(x) = -0.000006x^2 - 0.04x + 1000$$

$$\overline{C}'(x) = 0.000004x - 0.02 - 120,000x^{-2}$$

c. Compute $C'(3000)$, $R'(3000)$, and $P'(3000)$. and interpret your results.

$C'(3000) = 0.000006(3000)^2 - 0.04(3000) + 1000 = 934 = \text{approx. cost of producing the 3001st item}$

$R'(3000) = 2000 - 0.08(3000) = 1760 = \text{approx. revenue from producing the 3001st item}$

$P'(3000) = -0.000006(3000)^2 - 0.04(3000) + 1000 = 826 = \text{approx. profit from producing the 3001st item}$

4. The proprietor of the Showplace, a video club, has estimated that the rental price $p$ (in dollars) of prerecorded video cassette tapes is related to the quantity $x$ rented per week by the demand equation $x = \frac{1}{4} \sqrt{49 - p^2}$ where $(0 \leq p \leq 7)$

a. Find the elasticity when $p = $2.

$$f(p) = \frac{1}{4} \sqrt{49 - p^2}, \quad f'(p) = \frac{1}{4} \cdot \frac{1}{2} \left(49 - p^2\right)^{-\frac{1}{2}} (-2p) = -\frac{1}{4} \frac{p(49 - p^2)^{\frac{1}{2}}}{49 - p^2}$$

$$E = \frac{p}{\frac{1}{4} \sqrt{49 - p^2}} \cdot \frac{\frac{p^2}{49 - p^2}}{\frac{1}{4} \sqrt{49 - p^2}} = \frac{p^2}{49 - p^2}$$

at $p = 2$
b. Is the demand elastic or inelastic at this rental price? 
E = 0.0889 < 1 so the demand is inelastic

c. If the rental price is increased, will the revenue increase or decrease? When the price increases the demand increase to the revenue also increases.

d. Determine the interval where the demand will be unitary, the interval where demand will be inelastic, and the interval where it will be elastic.

Solve \( E = \frac{p^2}{(40 - p^2)} = 1 \), \( \frac{p^2}{(40 - p^2)} = 1 \rightarrow p^2 = 40 - p^2 \rightarrow 2p^2 = 40 \rightarrow p^2 = 20 \)
so \( p = \pm \sqrt{20} = \pm 2\sqrt{5} \), p is a price so we only use the positive root.

Now we need to check the value of E for p between our intervals of \( 0 \leq p < 2\sqrt{5} \) and \( 2\sqrt{5} < p \leq 7 \).
Unitary when \( p = 2\sqrt{5} \)
Inelastic for \( 0 \leq p < 2\sqrt{5} \)
Elastic for \( 2\sqrt{5} < p \leq 7 \)

e. In all of the above three cases, tell what happens the total revenue when the price increases.
Unitary: revenue does not change.
Inelastic: revenue increases/
Elastic: revenue decreases.

Answers Section 3.5; Higher-Order Derivatives

1. \( f'(x) = (x^2 + 1)(1 + 5x^2) \), 
   \( f''(x) = 4x(3 + 5x^2) \),

2. \( f'(x) = \frac{x^2 - 2x}{(x - 1)^2} = \frac{x(x - 2)}{(x - 1)^2} \) 
   \( f''(x) = \frac{2}{(x - 1)^3} \)
1. a. \( y = -(3/2)x + 5/2 \)

2. Suppose that the wholesale price of a certain brand of medium-size eggs \( p \) (in dollars per carton) is related to the weekly supply \( x \) (in thousands of cartons) by the equation \( 625p^2 - x^2 = 100 \). If 25,000 cartons of eggs are available at the beginning of a certain week and the price is falling at the rate of 2 cents per carton per week, at what rate is the supply falling?

If \( x = 25 \) then \( 625p^2 - (25)^2 = 100 \) implies \( p = 1.077 \), \( dp/dt = -.02 \)

so find \( \frac{d}{dt}(625p^2 - x^2 = 100) = 1250p \frac{dp}{dt} - 2x \frac{dx}{dt} = 0 \) now plug in your values and solve

for \( dx/dt \), \( 1250(1.077)(-.02) - 2(25) \frac{dx}{dt} = 0 \), so, \( \frac{dx}{dt} = -.5385 \), or 538.5 dozen cartons,

so the rate of supply is falling at approx 539 dozen eggs per week.

3. \( \frac{d}{dx} \left( \frac{x + y}{x - y} \right) = \frac{d}{dx} \left( x + y = 3x(x - y) \right) = 1 + y' = 3(x - y) + 3x(1 - y') \)

or \( y' = \frac{6x - 3y - 1}{1 + 3x} \)

4. \( \frac{d}{dx} \left( (2x + 3y)^{2/3} = x^2 \right) = \frac{1}{3} \left( 2x + 3y \right)^{-1/3} (2 + 3y') = 2x \)

or \( y' = \frac{6x(2x + 3y)^{-1/3} - 2}{3} \)

5. A 13-foot ladder is leaning against a wall. If the bottom of the ladder is pulled away from the wall at a rate of 2.5 ft/sec, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 12 feet from the wall.

From the Pythagorean theorem we know that for a right triangle \( x^2 + y^2 = l^2 \) where \( l \) is the length of the ladder, \( x \) is the distance from the bottom of the wall to the bottom of the ladder and \( y \) is the height of the ladder against the wall.

So when \( l = 13 \) and \( x = 12 \), \( y \) must be 5.

Differentiating our equation with respect to time we get

\[ 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2l \frac{dl}{dt} \]

but we know that \( x = 12, l = 13, y = 5 \) and \( dx/dt = 2.5 \) ft/sec
\( dl/dt = 0 \) (the length of the ladder does not change) and we are interested in finding \( dy/dt \).

so we have \( 2(12)(2.5) + 2(5) \frac{dy}{dt} = 2(13)(0) \), so \( dy/dt = -60/10 \) ft/sec = -6 ft/sec at that moment.
Answers Section 3.7: Differentials

1. Let \( f \) be a function defined by \( y = f(x) = \sqrt{2x + 1} \)
   a. Find the differential of \( f \).
      
      \[
      dy = f'(x) = \frac{1}{2} (2x + 1)^{\frac{1}{2}} \cdot 2dx = (2x + 1)^{\frac{1}{2}} dx
      \]
   
   b. Use your result from (a) to find the approximate change in \( y \) if \( x \) changes from 4 to 4.1
      
      \[
      dy = (2(4) + 1)^{\frac{1}{2}} (.1) = .033333
      \]
   
   c. Find the actual change in \( y \) if \( x \) changes from 4 to 4.1 and compare your result with that obtained in (b).
      \[
      f(4.1) - f(4) = \sqrt{2(4.1) + 1} - \sqrt{2(4) + 1} = .0331501776
      \]

2. Use differentials to approximate \( \sqrt{26.1} \) to two decimal places.
   
   Let \( f(x) = \sqrt{x} \) then \( f'(x) = \frac{1}{2\sqrt{x}} \)
   
   Then \( \Delta y \approx dy = f'(x)dx = \frac{1}{2\sqrt{x}} dx = \frac{1}{2\sqrt{25}} (1.1) = \frac{1.1}{10} = .11 \)
   
   so \( \sqrt{26.1} - \sqrt{25} = \Delta y \approx dy = .01 \), which means \( \sqrt{26.1} \approx \sqrt{25} + dy = 5 + .11 = 5.11 \)
   
   Note: \( \sqrt{26.1} = 5.10881591... \)