Math 23
McGraw
Simple Interest I = Prt A = P(1 + rt)

Compound Interest
A = P(1 + i)^n = P \left( 1 + \frac{r}{m} \right)^{mt}

Future Annuity
S = R \left[ \frac{(1+i)^n - 1}{i} \right] = R \left[ \frac{(1 + \frac{r}{m})^{mt} - 1}{\frac{r}{m}} \right]

Present Annuity
P = R \left[ 1 - (1+i)^{-n} \right] = R \left[ 1 - \frac{1}{\left( 1 + \frac{r}{m} \right)^{mt}} \right]

Section 5.1: Exponential Functions
1. Solve the following for x. 8^x = \left( \frac{1}{32} \right)^{x-2}
2. Solve the following for x. 5^{5x-2} = 125^x

Section 5.2: Logarithmic Functions
1. What is the solution for \frac{50}{1 + 4e^{0.2t}} = 20
   a. –4.9 \hspace{1cm} b. ln(3/8)/.2 \hspace{1cm} c. 0 \hspace{1cm} d. 4.904146 \hspace{1cm} e. None of these.
2. Use the log rules to expand the given expression \log \frac{x^2 \sqrt{3x - 4}}{(x - 5)^6}
3. Solve 12 + 4e^{-At} = 36

Section 5.3: Compound Interest
1. John got $500 for his birthday, he does not want to spend it, so he invested into an account that receives 7.5% compounded quarterly. If he does not plan to add any money or take any money out, how much will he have at the end of 5 years?
2. How much would you have to put into a savings account today, to have $3500 in 3 years to use as a down payment for a car. Assume you can get 5.58% interest compounded monthly.
3. How much money would you have to invest today to have $25,000 in the bank in 5 years if you are to receive 7.35% interest compounded continuously?
### Section 5.3b: Annuities

1. What is your monthly payment for a car loan of $21,500 over 4 years with an interest rate of 7.99% compounded monthly.
   
   a. $524.78  
   b. $1761.86  
   c. $381.62  
   d. $1727.00  
   e. None of these.

2. If you decide you can afford to put away into an Mutual Fund $250 a month, you know you can get 7.25% interest compounded monthly, how much will you have in the account in 30 years?
   
   a. $320,473.32  
   b. $36,647.42  
   c. $90,000  
   d. $2186.19  
   e. None of these.

### Section 5.3c: Amortization and Sinking Funds

1. Mr. Jones wants to buy a house for $350,000, he can make a down payment of $75,000 and he can take out a mortgage for the balance at 6.9% interest compounded monthly for the next 30 years, and his monthly payments are $1811.15. What part of his first payment is interest and what part is actually paying off the mortgage? How much does Mr. Jones actually pay for the house when he is done making payments? How much interest did he pay over the life of the loan? How much does he owe at the end of 10 years?

### Section 5.4: Differentiation of Exponential Functions

1. What is the derivative for $f(x) = (x-1)e^{3x+2}$
   
   a. $f'(x) = 3e^{3x+2}$  
   b. $f'(x) = (3x-2)e^{3x+2}$  
   c. $f'(x) = e^{3x+2}$  
   d. $f'(x) = 3(x-1)e^{3x+2}$  
   e. None of these.

2. Find the derivatives of the following: (Simplify your answers)
   
   a. $f(x) = \frac{e^{-x}}{1+x^2}$  
   b. $f(x) = (e^x + 1)^{25}$

3. Determine the intervals where the function $f(x) = x^2 e^{-x}$ is increasing and decreasing (you should also be able to find the intervals of concavity and be able to graph the function).

4. Find the equation of the line tangent to the graph of $y = e^{-x^2}$ at the point (1, 1/e)

5. Find the absolute extrema for $f(x) = e^{x^2-4}$ on [-2, 2]
### Section 5.5: Differentiation of Logarithmic Functions

1. Find the second derivative of $f(x) = (\ln x)^2$

- a. $f''(x) = \frac{2 \ln(x)}{x}$
- b. $f''(x) = \frac{1}{x}$
- c. $f''(x) = \frac{2 \ln(x)}{x^2}$
- d. $f''(x) = \frac{2 - 2 \ln(x)}{x^2}$
- e. None of these.

2. Find the derivatives of the following: (Simplify your answers)

- a. $f(x) = \ln \left( \frac{2x}{x+1} \right)$
- b. $f(x) = x^2 \ln(e^{2x} + 1)$

3. Find the intervals of concavity for $f(x) = x^2 + \ln x^2$

(you should also be able to find the intervals where $f$ is increasing and decreasing and be able to graph the function).

4. Find the absolute extrema for $f(x) = \frac{x}{\ln x}$ on $[2, \infty)$.

### Section 5.6: Exponential Functions as Mathematical Models

1. Skeletal remains of the so-called “Pittsburgh Man” unearthed in Pennsylvania, had lost 82% of the Carbon-14 they originally contained. Determine the approximate age of the bones. Recall Carbon –14 has a half life of 5770 years.

2. On the basis of data collected during an experiment, a biologist found that the growth of the fruit fly (Drosophila) with a limited food supply could be approximated by the exponential model $N(t) = \frac{400}{1 + 39e^{-0.16t}}$ where $t$ denotes the number of days since the beginning of the experiment.

   a. What is the initial fruit-fly population in the experiment?
   b. What was the maximum fruit-fly population that could be expected under this laboratory condition?
   c. What was the population of the fruit-fly colony on the twentieth day?
### Answers Section 5.1: Exponential Functions

1. Solve the following for \( x \).
   \[ 8^x = \left( \frac{1}{32} \right)^{x-2} \]
   Notice we really have \((2^3)^x = (2^{-5})^{x-2}\), so \(3x = -5(x-2)\) or \(x = 5/4\)

2. Solve the following for \( x \).
   \[ 5^{5x-2} = 125^{6x} \]
   \[ 5^{5x-2} = (5^3)^{6x} \]
   \[ 5x - 2 = 3(6x) \]
   \[ 5x - 2 = 18x \]
   \[ -2 = 13x \]
   \[ x = \frac{-2}{13} \]

### Answers Section 5.2: Logarithmic Functions

1. What is the solution for
   \[ \frac{50}{1 + 4e^{0.2t}} = 20 \]
   \[ 50 = 20(1 + 4e^{0.2t}) \]
   \[ \frac{5}{2} = 1 + 4e^{0.2t} \]
   \[ \frac{3}{2} = 4e^{0.2t} \]
   \[ \frac{3}{8} = e^{0.2t} \]
   \[ \ln \left( \frac{3}{8} \right) = \ln e^{0.2t} = 0.2t \]
   \[ \frac{\ln(3/8)}{0.2} = t \]
   b. \( \ln(3/8) / 0.2 \)

2. Use the log rules to expand the given expression
   \[ \log \frac{x^2 \sqrt{3x-4}}{(x-5)^{10}} = \log x^2 + \log \sqrt{3x-4} - \log(x-5)^{10} = 2 \log x + \frac{1}{2} \log(3x-4) - 10 \log(x-5) \]
3. Solve
12 + 4e^{-4t} = 36
4e^{-4t} = 24
e^{-4t} = 6
-.4t = \ln(6)
t = \ln(6)/(-.4) = -4.479398673

Answers Section 5.3
1. John got $500 for his birthday, he does not want to spend it, so he invested into an account that receives 7.5% interest per quarter. If he does not plan to add any money or take any money out, how much will he have at the end of 5 years?

\[ A = 500 \left( 1 + \frac{.075}{4} \right)^{(5)(4)} = 724.97 \]

2. How much would you have to put into a savings account today, to have $3500 in 3 years to use as a down payment for a car. Assume you can get 5.58% interest compounded monthly.

\[ 3500 = P \left( 1 + \frac{.0558}{12} \right)^{12(3)} \]
\[ 3500 = P(1.591046228) \]
\[ P = 2961.66 \]

3. How much money would you have to invest today to have $25,000 in the bank in 5 years if you are to receive 7.35% interest compounded continuously?

\[ 25,000 = Pe^{0.0735(5)} \]
\[ P = \frac{25000}{e^{0.3675}} = 17311.58 \]

Answers Section 5.3b: Annuities
1. What is your monthly payment for a car loan of $21,500 over 4 years with an interest rate of 7.99% compounded monthly. a. $524.78

2. If you decide you can afford to put away into an Mutual Fund $250 a month, you know you can get 7.25% interest compounded monthly, how much will you have in the account in 30 years? a. $320,473.32
1. Mr. Jones wants to buy a house for $350,000, he can make a down payment of $75,000 and he can take out a mortgage for the balance at 6.9% interest compounded monthly for the next 30 years, and his monthly payments are $1811.15. What part of his first payment is interest and what part is actually paying off the mortgage? How much does Mr. Jones actually pay for the house when he is done making payments? How much interest did he pay over the life of the loan?

He needs a mortgage for $350,000 – $75,000 = $275,000.

So the interest paid for any given month =
(\text{previous months unpaid balance}) \times (\text{monthly interest rate})

for the first month he owes the total amount, so
interest of first month = 275,000 \times (0.069/12) = 1581.25 which is the part of the first month’s payment that is interest.

so $1,811.15 – $1,581.25 = $229.9 = how much of the first payment that actually goes toward paying off the house.

$1,811.15 \times 12 \times 30 = $652,014 how much the payments add up to, so he paid $652,014 + $75,000 = $727,014 for the house.

His total interest paid is $652,014 – $275,000 = $377,014.

At the end of 10 years he has made 120 payments, so he has 360 – 120 left, or 240 more payments, so find the present value of 180 payments of $1811.15

or \[ P = 1811.15 \left[ \frac{1 - (1 + 0.069/12)^{-240}}{0.069/12} \right] = 235425.92 \]
left to pay on the house.

Answers Section 5.4: Differentiation of Exponential Functions

1. What is the derivative for \( f(x) = (x-1)e^{3x+2} \) \( f'(x) = (3x-2)e^{3x+2} \)

2. a. \( f(x) = \frac{e^{-x}}{1+x^2} \) use quotient rule to get

\[ f'(x) = \frac{-e^{-x}(1 + x^2) - e^{-x}(2x)}{(1+x^2)^2} = \frac{-e^{-x}(x^2 + 2x + 1)}{(1+x^2)^2} = \frac{-e^{-x}(x+1)(x+1)}{(1+x^2)^2} \]

b. \( f(x) = (e^x + 1)^{25} \) use chain rule to get

\[ f''(x) = 25(e^x + 1)^{24} e^x = 25e^x(e^x + 1)^{24} \]
3. Determine the intervals where the function \( f(x) = x^2e^{-x} \) is increasing and decreasing. 
\[
f'(x) = 2xe^{-x} + x^2e^{-x}(-1) = e^{-x}(2x - x^2)
\]
this is always defined and will only equal zero when \((2x - x^2) = x(2 - x) = 0\) or when \(x = 0\) or \(x = 2\).

\[
\begin{array}{|c|c|}
\hline
x & f'(x) \\
\hline
-1 & e^{-(-1)}(2(-1) - (-1)^2) = -8.15 \\
1 & e^{(-1)}(2(1) - (1)^2) = .368 \\
4 & e^{(-4)}(2(4) - (4)^2) = -.1465 \\
\hline
\end{array}
\]

So \( f \) is increasing on \((0, 2)\) and decreasing on \((-\infty, 0)\) and \((2, \infty)\) 
There is a relative min at \(x = 0\) and a relative max at \(x = 2\).

(you should also be able to find the intervals of concavity and be able to graph the function).

\[
** f''(x) = -e^{-x}(2x - x^2) + e^{-x}(2 - 2x) = e^{-x}(x^2 - 4x + 2)
\]

Setting \( f'' = 0 \) yields \( 2 \pm \sqrt{2} \), Check for changes in concavity to find that 
\( f \) is concave up \((-\infty, 2 - \sqrt{2}) \cup (2 + \sqrt{2}, \infty)\) and concave down \((2 - \sqrt{2}, 2 + \sqrt{2})\)

There are no vertical asymptotes, but there is a horizontal one when \(y = 0\) (the x-axis).

4. Find the equation of the line tangent to the graph of \( y = e^{-x^2} \) at the point \((1, 1/e)\)
\( y' = -2xe^{-x^2} \) at \(x = 1\) the slope is \( y' = -2(1)e^{-1} = -2e^{-1} = \frac{-2}{e} \)

to the equation of the tangent line is 
\[
y - \frac{1}{e} = -\frac{2}{e}(x - 1) \text{ or } y = -\frac{2}{e}x + \frac{3}{e}
\]

5. Find the absolute extrema for \( f(x) = e^{x^2-4} \) on \([-2, 2]\)
\( f'(x) = 2xe^{x^2-4} \) which will only be zero at \(x = 0\) (\( f' \) is always defined)

\[
\begin{array}{|c|c|}
\hline
x & f(x) \\
\hline
-2 & e^0=1 \\
0 & e^{-4}=0.0183156389 \\
2 & e^0=1 \\
\hline
\end{array}
\]

so \( f(0) = e^{-4} \) is the absolute minimum 
and \( f(2) = f(-2) = 1 \) are the absolute maximums
### Answers Section 5.5: Differentiation of Logarithmic Functions

1. Find the second derivative of \( f(x) = (\ln x)^2 \)  

d. \( f''(x) = \frac{2 - 2\ln(x)}{x^2} \)

2. a. \( f(x) = \ln\left(\frac{2x}{x+1}\right) = f(x) = \ln(2x) - \ln(x+1) \) by using log rules  

so \( f'(x) = \frac{1}{2x} \cdot 2 - \frac{1}{x+1} = \frac{1}{x(x+1)} \)

b. \( f(x) = x^2 \ln(e^{2x} + 1) \) use product rule to get  

\[
f'(x) = 2x \ln(e^{2x} + 1) + x^2 \frac{1}{e^{2x} + 1} 2e^{2x} = \frac{2x(e^{2x} + 1) \ln(e^{2x} + 1) + 2x^2 e^{2x}}{e^{2x} + 1}
\]

3. Find the intervals of concavity for \( f(x) = x^2 + \ln x^2 \)

\( f'(x) = 2x + \frac{2}{x} \)

\( f''(x) = 2 - \frac{2}{x^2} \) the second derivative is undefined at \( x = 0 \) and equals zero when \( \frac{2}{x^2} = 0 \) or \( 2x^2 - 2 = 0 \), so \( x = \pm 1 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f''(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>( 2 - \frac{2}{(-2)^2} = 1.5 )</td>
</tr>
<tr>
<td>-.5</td>
<td>( 2 - \frac{2}{(-.5)^2} = -6 )</td>
</tr>
<tr>
<td>.5</td>
<td>( 2 - \frac{2}{(.5)^2} = -6 )</td>
</tr>
<tr>
<td>2</td>
<td>( 2 - \frac{2}{(2)^2} = 1.5 )</td>
</tr>
</tbody>
</table>

So \( f \) is concave up on \(( -\infty, -1 ) (1, \infty) \) and concave down on \((- 1, 0) \) and \((0, 1) \)  

(you should also able to find the intervals where \( f \) is increasing and decreasing and be able to graph the function).

** set \( f'(x) = 2x + \frac{2}{x} = 0 \), to get \( 2x^2 + 2 = 0 \) which yields no solutions, but \( f' \) is undefined at \( x = 0 \), so this is our only critical value. \( f \) is increasing on \((0, \infty) \) and \( f \) is decreasing on \((-\infty, 0) \). \( f \) is undefined at \( x = 0 \), there are no horizontal asymptotes, but there is vertical on at \( x = 0 \).
4. Find the absolute extrema for \( f(x) = \frac{x}{\ln x} \) on \([2, \infty)\).

\[
f'(x) = \frac{1}{x} - \frac{1}{x \ln x} = \frac{\ln(x) - 1}{(\ln x)^2},
\]

\( f' \) is undefined for \( x < 0 \) and \( x = 1 \) because \( \ln(1) = 0 \).

and \( f'(x) = 0 \) when \( \ln(x) = 1 \) or when \( \ln(x) = 1 \) so \( x = e \),

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( \frac{2}{\ln 2} \approx 2.88 )</td>
</tr>
<tr>
<td>( e )</td>
<td>( \frac{e}{\ln e} = e \approx 2.718 )</td>
</tr>
<tr>
<td>( \infty )</td>
<td>( \lim_{a \to \infty} \frac{a}{\ln a} \approx \infty )</td>
</tr>
</tbody>
</table>

looks like and absolute min is at \((e, e)\) and no absolute max because \( f(x) \) increases without bound after \( x = e \), to be sure look at the first derivative test for \( f(x) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( \frac{\ln(2) - 1}{(\ln 2)^2} \approx -0.63 )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{\ln(3) - 1}{(\ln 3)^2} \approx 0.081 )</td>
</tr>
</tbody>
</table>

so \( f(x) \) decreases before the critical value \( x = e \) and increases after \( x = e \).

so \((e, e)\) is the only absolute extrema and it is a minimum, no absolute max.
Section 5.6: Exponential Functions as Mathematical Models

1. Skeletal remains of the so-called “Pittsburgh Man” unearthed in Pennsylvania, had lost 82% of the Carbon-14 they originally contained. Determine the approximate age of the bones. Recall Carbon –14 has a half life of 5770 years.

\[ \frac{1}{2} Q_0 = Q_0 e^{-5770k} \]

\[ \frac{1}{2} = e^{-5770k} \]

\[ \ln \left( \frac{1}{2} \right) = -5770k \]

\[ k \approx 0.000120 \]

Lost 82% so 18% is left, \( Q_0 \) is the initial amount.

\[ Q(t) = 0.18Q_0 = Q_0 e^{-0.00012t} \]

\[ 0.18 = e^{-0.00012t} \]

\[ \ln(0.18) = -0.00012t \]

\[ t \approx 14289.9869 \]

2. On the basis of data collected during an experiment, a biologist found that the growth of the fruit fly (Drosophila) with a limited food supply could be approximated by the exponential model \( N(t) = \frac{400}{1 + 39e^{-0.16t}} \) where \( t \) denotes the number of days since the beginning of the experiment.

a. What is the initial fruit-fly population in the experiment?

\[ N(0) = \frac{400}{1 + 39e^{-0.16(0)}} = \frac{400}{1 + 39} = \frac{400}{40} = 10 \]

b. What was the maximum fruit-fly population that could be expected under this laboratory condition?

\[ \lim_{t \to \infty} \frac{400}{1 + 39e^{-0.16t}} = \lim_{t \to \infty} \frac{400}{1 + \frac{39}{e^{0.16t}}} = 400 \]

c. What was the population of the fruit-fly colony on the twentieth day?

\[ N(20) = \frac{400}{1 + 39e^{-0.16(20)}} = 154.46 \]