1) Complete:
   a) If \( A \) is a 3x5 matrix then \( \text{rank} A \) is at most _________.
   b) If \( A \) is a 3x5 matrix then \( A \) ________ (can or cannot) represent a one to one linear transformation.
   c) If \( T : V \rightarrow W \) is a linear transformation between finite dimensional vector spaces then
      i) The domain of \( T \) is __________
      ii) The codomain of \( T \) is __________
      iii) ________, \( \text{dim} (\ker T) + \text{dim} (\text{Range} T) = \) __________
   d) The \( \vec{0} \) for \( C[0,1] \) using standard operations is __________

2) The statements below are either properties/theorems that you were supposed to learn or they are incomplete statements of properties/theorems you were supposed to learn. If they are correctly stated write "complete". If they are not quite completely stated write "not quite" and then write the correct complete statement down.
   a) Let \( \{y_1, y_2, \cdots, y_n\} \) be a set of functions each of which has \( n-1 \) derivatives on the interval \( I \). This set is linearly independent if and only if the Wronskian is not identically zero.
   b) Given \( S = \{p_1, p_2, \cdots, p_k\} \) a spanning set of a vector space \( V \). If \( p_k \in \text{Span} S \) then
      \( \{p_1, p_2, \cdots, p_{k-1}\} \) spans \( V \) too.
   c) \( \dim P_6 = 6 \)
   d) "additive identity axiom for vector spaces" states "For every \( \mathbf{u} \in V \), \( \mathbf{u} + \mathbf{0} = \mathbf{u} \)."

3) Given \( S = \{p_1, p_2, p_3, p_4\} \subseteq P_4 \) where
   \[
   p_1(x) = x^4 + 3x^3 + 2x + 4, \\
   p_2(x) = x^3 - x^2 + 5x + 1, \\
   p_3(x) = x^4 + x^3, \text{ and} \\
   p_4(x) = x^4 + x^3 - x + 2.
   \]
   Determine whether the set \( S \) is linearly independent or linearly dependent.
4) Given \( A = \begin{pmatrix} 1 & -3 & 4 & -2 & 5 \\ 2 & -6 & 9 & -1 & 8 \\ 2 & -6 & 9 & -1 & 9 \\ -1 & 3 & -4 & 2 & -5 \end{pmatrix} \). Consider the linear transformation defined by \( A \): \( T(x) = Ax \).

Find the rank of \( T \), the nullity of \( T \), a basis for the kernel (null space), and a basis for the range.

5) Given the basis \( B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} \right\} \) for \( M_{2,2} \).

If \( A = \begin{pmatrix} 7 \\ -3 \\ 3 \\ -1 \end{pmatrix} \) find \( [A]_B \).

6) In class homework we proved that \( V = \mathbb{R}^3 \) together with the nonstandard operations:

\[
(x, y, z) + (a, b, c) = (x + a + 1, y + b + 1, z + c + 1)
\]

\[
k(x, y, z) = (kx + k - 1, ky + k - 1, kz + k - 1)
\]

forms a vector space.

Let \( T : \mathbb{R}^2 \to V \) be a linear transformation. If \( T(1, 0) = (3, 5, 1) \) and \( T(0, 1) = (2, 4, 0) \) find \( T(-2, 3) \).

(Don’t forget we are using nonstandard operations in \( V! \))

7) Given \( T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3z - 2y \\ 4x + 11z \end{pmatrix} \)

a) What is the domain of \( T \)?

b) What is the codomain of \( T \)?

c) Determine the standard matrix for the linear transformation.

d) Is \( T \) an onto transformation? Why or why not?

8) Let \( W = \{ x = (x, y, z, w) \mid 2x - y + 5z = 0 \} \).

a) Show \( W \) is a subspace of \( \mathbb{R}^4 \).

b) Determine a basis for \( W \) and state the dimension of \( W \).

9) Given \( T : V \to W \) is a linear transformation between finite dimensional vector spaces. If \( \{Tv_1, Tv_2, \ldots, Tv_k\} \) is an independent set in \( W \), then prove \( \{v_1, v_2, \ldots, v_k\} \) is an independent set in \( V \).

(Yes this is exactly your homework from 6.1)
Answers

1) 
   a) Notice matrix takes $\mathbb{R}^5$ to $\mathbb{R}^3$ and rank is at most the min(of number of columns and rows) so 3.
   b) Cannot since it takes dim 5 space to dim 3 space. (It could be onto.)
   c) 
      i) V
      ii) W
      iii) Dim V
   d) $f$ where $f(x) = 0$ for all $x \in [0,1]$

2) 
   a) “not quite” Can fix in three ways. Way 1: Add “these functions are solutions to a nth order homogeneous differential equation” Way 2: change it to “functions dependent then wronskian is always zero.” Way 3: change it to “if wronskian is not identically zero then functions are independent,”
   b) complete problem 52 section 4.4 and discussed in class
   c) “not quite” dimension is seven
   d) “not quite” need to add “there exists a vector in V called 0

3) Using coordinate vectors with respect to standard basis.

\[
[p_1] = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix},
[p_2] = \begin{pmatrix} 0 \\ -1 \\ 5 \\ 1 \end{pmatrix},
[p_3] = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \end{pmatrix},
[p_4] = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 2 \end{pmatrix},
[0] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]

Consider $\sum c_i[p_i] = [0]$. Only solution is trivial solution so independent

4) Since $rref(A) = \begin{pmatrix} 1 & -3 & 0 & -14 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ has three pivots we have rank T=3. Since there are two columns without pivots nullity of T=2. To find a basis for kernel T you must solve $T(x) = Ax = 0$. Notice x is a column vector. So from rref(A) you find $x_1 - 3x_2 - 14x_4 = 0, x_3 + 3x_4 = 0, x_5 = 0$. Solving these for $x_1, x_3, x_5$ and substituting into $x^T = (3x_2 + 14x_4, x_2, -3x_4, x_4, 0)^T$. Thus \{(3,1,0,0,0)^T, (14,0,-3,1,0)^T\} forms a basis for ker T. For range T since column 1,3, and 5 have pivots can use \{p_1, p_3, p_5\} for a basis of Range T.

5) You have to solve $\sum c_i v_i = A$ where the v’s are the basis vectors. You can use coordinate vectors.

Your answer is a column whose entries are the c’s.

\[
\begin{pmatrix} 16 \\ -5 \\ -1 \\ -1 \end{pmatrix}
\]
6) \[ T(-2,3) = T(-2(1,0) + 3(0,1)) = -2T(1,0) + 3T(0,1) = (-9,-13,-5) + (8,14,2) = (0,2,-2) \]

7) a) \( \mathbb{R}^3 \)  
   b) \( \mathbb{R}^2 \)  
   c) \( \begin{pmatrix} 0 & -2 & 3 \\ 4 & 0 & 11 \end{pmatrix} \) Use standard basis so \( v_i \) has a “1” in ith position 0’s else. Col i is \( T(v_i) \)  
   d) Since \( \text{Rank}T=2=\text{dim of codomain} \) it is onto.

8) Notice \( W = \{ (x,y,z,w) \mid (2 -1 5 0)x^T = 0 \} = \ker \begin{pmatrix} 2 & -1 & 5 & 0 \end{pmatrix} \) so it is a vector space. Also notice \( x = (x,y,z,w) = (x,2x+5z,z,w) \) so basis is \((1,2,0,0), (0,5,1,0), (0,0,0,1)\)

9) Proof: Given \( T \) is a linear transformation. Assume \( \sum d_i v_i = 0 \) Then apply the linear transformation to both sides using property of linearity and \( T(0) = 0 \) to obtain \( \sum d_iTv_i = 0 \). We are given that \( \{Tv_i : 1 \leq i \leq k\} \) form a linear independent set therefore we conclude that the scalars \( d_i = 0 \). Thus \( \{v_i : 1 \leq i \leq k\} \) forms a linear independent set.