Physics Laboratory  
Gauss’s Law

I. Electric flux through closed surfaces

We found that the electric flux through a set of imaginary surfaces, \( d\vec{A}_i \), each with an electric field, \( \vec{E}_i \), can be written as:

\[
\Phi_E = \vec{E}_1 \cdot d\vec{A}_1 + \vec{E}_2 \cdot d\vec{A}_2 + \ldots = \int \vec{E} \cdot d\vec{A}
\]

The area vectors at each point on a closed surface (i.e., a surface that surrounds a region so that the only way out of the region is through the surface) are chosen by convention to point out of the enclosed region. A closed imaginary surface is called a Gaussian surface.

The diagram to the right shows a Gaussian surface in the shape of a right circular cylinder with a radius \( R \) and a height \( H \). The top surface is labeled A, the cylindrical side surface is labeled B, and the bottom surface is labeled C.

A. The Gaussian surface is in a uniform electric field of magnitude, \( E_0 \), that is aligned with the cylinder axis.

1. Find the sign and magnitude of the flux through:

   Surface A: \( + E_0 R^2 \)  
   Surface B: 0  
   Surface C: \( -E_0 R^2 \)

2. Is the net flux through the Gaussian surface positive, negative, or zero? Explain.

   \[ \text{zero if the number of flux lines entering is the same as the number of flux lines leaving} \]

B. The Gaussian surface now encloses a negative charge. (The field from part A is removed.)

1. Find the sign of the flux through

   Surface A:  
   Surface B:  
   Surface C: 

2. Is the net flux through the Gaussian surface positive, negative, or zero? Explain.

   \[ \text{negative, all the flux lines are entering the surface} \]
C. The Gaussian surface now encloses opposite charges of equal magnitude. (The charges are on the axis of the cylinder and equidistant from the center.)

1. Find the sign of the flux through

\[
\begin{array}{ccc}
\text{Surface A:} & \text{Surface B:} & \text{Surface C:} \\
+ & 0 & -
\end{array}
\]

2. Is the net flux through the Gaussian surface positive, negative, or zero?

zero.

D. A positive charge is located above the Gaussian surface.

1. Find the sign of the flux through

\[
\begin{array}{ccc}
\text{Surface A:} & \text{Surface B:} & \text{Surface C:} \\
- & + & +
\end{array}
\]

2. Is the net flux through the Gaussian surface positive, negative, or zero? Explain.

zero. Number of flux lines entering the surface is the same as the number leaving the surface.

II. Gauss’s law

Gauss’s Law \( \Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{enclosed}}}{\varepsilon_0} \) states that the electric flux through a Gaussian surface is directly proportional to the net charge enclosed by the surface.

A. Are your answers to parts A, B, and C of section I consistent with Gauss’s law? Explain.

A. \( \Phi_E = 0 \) because \( q_{\text{end}} = 0 \)

B. \( \Phi_E = 0 \) because \( q_{\text{end}} = Q \)

B. In part D of section I, you determined the sign of the flux through the surfaces A, B, and C of the cylindrical Gaussian surface. If \( \Phi_A = -10 \text{ N m}^2/\text{C} \) and \( \Phi_C = +2 \text{ N m}^2/\text{C} \), what is \( \Phi_B \)?

\[
\Phi_E = \Phi_A + \Phi_B + \Phi_C = 0
\]

\[
= -10 + \Phi_B + 2 = 0
\]

\[
\Phi_B = 8 \text{ N m}^2/\text{C}
\]
C. Find the net flux through each of the cylindrical Gaussian surfaces below.

1. \( \Phi_e = 0 \)
2. \( \Phi_e = \frac{Q_0}{\varepsilon_0} \)
3. \( \Phi_e = -\frac{Q_0}{\varepsilon_0} \)
4. \( \Phi_e = -\frac{Q_0}{\varepsilon_0} \)

D. The three Gaussian surfaces to the right each enclose a charge \( +Q_0 \). In case C there is another charge \( -6Q_0 \) outside the surface.

Consider the following conversation:

Student 1: "Since each Gaussian surface encloses the same charge, the net flux through each must be the same."

Student 2: "Gauss's law doesn't apply here. The electric field at the Gaussian surface in case B is weaker than in case A, because the surface is farther from the charge. Since the flux is proportional to the electric field strength, the flux must also be smaller in case B."

Student 3: "I was comparing A and C. In C the charge outside changes the field over the whole surface. The areas are the same, so the flux must be different."

Do you agree with any of the students? Explain.

Student 1: The flux only depends on the net charge enclosed by the surface. In the three cases, the enclosed charge is the same, so the flux must be the same.
III. Applications of Gauss's law

A. A large aluminum sphere of radius $b$ has a spherical cavity of radius $a$ inside it. A charge of $-2Q_0$ is placed at the center of the cavity.

1. What is the magnitude and direction of the electric field in the region between radii $a$ and $b$? Explain.

   $E = 0$, the region is a conductor.

2. How much charge is located on the inside and outside surfaces of the aluminum sphere? Explain your reasoning.

   a. inside surface

      By taking a Gaussian surface just inside the conductor, the flux must be zero because $E = 0$. The net charge enclosed is zero. There is a charge $-2Q_0$ in the cavity, so there must be a charge $+2Q_0$ on the inner surface of the conductor to make the charge enclosed equal to zero.

   b. outside surface

      A Gaussian surface just outside the sphere must have a net charge $-2Q_0$ enclosed. There is $-2Q_0$ in the cavity, +2Q on the inside surface of the conductor, so another $-2Q_0$ must be on the outside surface to give a total of $-2Q_0$.

3. If an additional amount of charge $+5Q_0$ is placed on the aluminum sphere, then how much charge is on the inside and outside surface of the aluminum sphere?

   a. inside surface

      Still $+2Q_0$.

   b. outside surface

      $+3Q_0$.
B. A long cylindrical rod of radius $R$ has a uniform charge density $\rho$ distributed throughout its volume.

1. What is the linear charge density ($\lambda = Q/L$) of the rod? Express your result in terms of $\rho$, $R$, and constants. *(Hint: Consider a length $L$ of the rod and the amount of charge in that length.)*

$$Q = \lambda L = \rho \pi R^2 L$$

$$\lambda = \rho \pi R^2$$

2. Use Gauss's Law to find the magnitude of the electric field at a distance $r$ outside the rod, where $r$ is measured radially outward from the axis of the rod.

$$\oint E \cdot dA = \frac{Q_{\text{enc}}}{\epsilon_0}$$

SetUp $\Phi_1$.

$$0 + 0 + E_{2\pi rL} = \lambda L$$

$$E = \frac{\lambda}{2\pi \epsilon_0 r} = \frac{\rho \pi R^2}{2\pi \epsilon_0 r} = \frac{\rho R^2}{2 \epsilon_0 r}$$

3. Use Gauss's Law to find the magnitude of the electric field inside the rod at a radial distance $r$ from the axis of the rod.

$$\oint E \cdot dA = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$E_{2\pi rL} = \frac{\rho \pi r^2 L}{\epsilon_0}$$

$$E = \frac{\rho L}{2\epsilon_0}$$
4. Extra: Repeat parts 2 and 3 to this problem if the charge density of the rod is not constant, but is a function of the distance from the axis of the rod, $\rho = Cr$, where $C$ is a constant and $r$ is the radial distance from the axis of the rod.

![Diagram of a cylindrical shell with a rod inside and outside.](image)

\[ \text{need to find:} \quad Q \text{ and } E \]

\[ \rho = \frac{dQ}{dV} = Cr, \quad \rho \neq \text{constant.} \]

\[ dQ = \rho dV = Cr dr \]

\[ dV = \text{thin cylindrical shell}, \]

\[ = 2\pi r dr L \]

\[ \Rightarrow dQ = Cr 2\pi r dr L \]

\[ Q = 2\pi CL \int_0^L r dr = 2\pi CL \left( \frac{R^3}{3} \right) \]

outside: \[ Q = 2\pi CL \left( \frac{R^3}{3} \right) \bigg|_0^R = \frac{2\pi\pi CR L R^3}{3} \]

\[ E 2\pi r L = \frac{2\pi\pi CR L R^3}{3} \]

\[ E = \frac{CR}{3\varepsilon_0 r} \]

inside: \[ Q = 2\pi CL \left( \frac{R^3}{3} \right) \bigg|_0^r = \frac{2\pi\pi CR L r^3}{3} \]

\[ E 2\pi r L = \frac{2\pi\pi CR L r^3}{3} \]

\[ E = \frac{CR^2}{3\varepsilon_0} \]