Homework Help - 3.1, 3.2, 3.3

3.1, #15) Solve: \( y'' + 8y' - 9y = 0, \quad y(1) = 1, \quad y'(1) = 0 \)

Characteristic eqn \( r^2 + 8r - 9 = 0 \) \( \Rightarrow \) \( r = -9, 1 \) \( \Rightarrow \) General solution is

\[ y = c_1 e^{-9t} + c_2 e^t \]

So \( y' = -9c_1 e^{-9t} + c_2 e^t \)

\( y(1) = 1 \) \( \Rightarrow \) \( 1 = c_1 e^{-9} + c_2 e^9 \) \( \text{(1)} \)

\( y'(1) = 0 \) \( \Rightarrow \) \( 0 = c_1 e^{-9} - 9c_2 e^9 \) \( \text{(2)} \)

Solving for \( c_1, c_2 \) gives

\[ \begin{align*}
9(\text{(1)}) + (\text{(2)}) & \quad 9 = 10c_1 e^9 \quad \Rightarrow \quad c_1 = \frac{9}{10} e^{-9} \\
(\text{(1)}) - (\text{(2)}) & \quad 10c_1 e^{-9} - c_2 = 10c_1 e^{-1} - c_2 \\
& \quad \Rightarrow \quad c_2 = \frac{1}{10} e^{-9}
\end{align*} \]

\( y = \frac{9}{10} e^{-9} e^t + \frac{1}{10} e^9 e^{-9t} \)

\( y = \frac{9}{10} e^{(t-1)} + \frac{1}{10} e^{9-9t} \) \( a_0 \) \( t \to \infty, \) \( t-1 \to -\infty, \) \( 9-9t \to -\infty \)

\( \Rightarrow \) \( y \to +\infty \to 0 \to +\infty \)

3.2, #1) \( W(\cos^2 \theta, 1 + \cos 2\theta) = \begin{vmatrix} \cos^2 \theta & 1 + \cos 2\theta \\ -2\cos \theta \sin \theta & -2 \sin 2\theta \end{vmatrix} \)

Note: \( \cos 2\theta = \cos^2 \theta - \sin^2 \theta \) so \( 1 + \cos 2\theta = 1 + \cos^2 \theta - \sin^2 \theta \)

\( = (1 - \sin^2 \theta) + \cos^2 \theta \)

\( = 2 \cos^2 \theta \)

\( \Rightarrow \sin 2\theta = 2 \sin \theta \cos \theta \)

Use \( \sin \theta \) and \( \cos \theta \) to rewrite \( W \):

\( W(\cos^2 \theta, 1 + \cos 2\theta) = \begin{vmatrix} \cos^2 \theta & 2\cos^2 \theta \\ -2\cos \theta \sin \theta & -4 \cos^2 \theta \sin \theta \end{vmatrix} \)

\( = -4 \cos^2 \theta \sin \theta + 4 \cos^2 \theta \sin \theta \)

\( = 0 \)
3.2. (e) Can \( y = \sin(t^2) \) be a sol'n on interval containing \( t = 0 \) to
\[
y'' + p(t) y' + q(t) y = 0. \quad \text{Suppose } y = \sin(t^2) \text{ is a solution on } (a, b) \text{ where } a < 0 < b.
\]
\[
y' = 2t \cos(t^2) \quad y'' = 4t \cos(t^2) - 4t^2 \sin(t^2)
\]
Then
\[
y'' + p(t) y' + q(t) y = 2t \cos(t^2) - 4t^2 \sin(t^2) + p(t) 2t \cos(t^2) + q(t) \sin(t^2)
\]
At \( t = 0 \), we have
\[
\frac{2 \cos(0) - 4(0) \sin(0) + p(0) 2(0) \cos(0) + q(0) \sin(0)}{2 - 0 + 0 + 0} = 2.
\]
Because the solution \( y = \sin(t^2) \) doesn't work at \( t = 0 \),
it doesn't work throughout the interval \((a, b)\). It may work on \((0, a)\) or \((0, b)\) or both but we can't tell without more information about \( p(t) \) and \( q(t) \).

3.2. (f) If \( W(f, g) = t^2 e^t \) and \( f(t) = t \), find \( g(t) \).

\[
W(f, g) = \begin{vmatrix} t & g \\ 1 & g' \end{vmatrix} = tg' - g = t^2 e^t \quad \text{if } t \neq 0,
\]
\[
g' - \frac{g}{t} = e^t \quad \text{1st ORDER LINEAR DE}
\]

We can solve for \( g \) using **INTERGRATING FACTOR** as in 2.1.
First refer to #27. \( P(x)y'' + (Q(x)y' + R(x))y = 0 \) is exact if it can be written as \([P(x)y']' + [f(x)y]' = 0\) where \(f(x)\) is determined based on \(P(x)\), \(Q(x)\) and \(R(x)\). We can conclude an equation is exact if \(P''(x) - Q'(x) + R(x) = 0\).

To check \(y'' + xy' + y = 0\) for exactness, compare to
\[
\begin{align*}
\begin{cases}
P(x)y'' + Q(x)y' + R(x)y = 0 \\
\begin{cases} P(x) = 1 & Q(x) = x & R(x) = 1 \\
Q'(x) = 0 & f(x) = 1 \\
P''(x) = 0
\end{cases}
\end{cases}
\end{align*}
\]

.: Equation is exact.

To solve, we need to find a function \(f(x)\) which enables use to rewrite the equation \(P(x)y'' + Q(x)y' + R(x)y = 0\) in form \([P(x)y']' + [f(x)y]' = 0\). This means by (Bernoulli), \(P(x)y'' + P(x)y' + f(x)y' + f(x)y = 0\) or \(P(x)y'' + [P'(x) + f(x)]y' + f(x)y = 0\) we have
\[
y'' + xy' + y = 0
\]
which gives \(P(x) = 1\), \(P'(x) + f(x) = x\) and \(f'(x) = 1\).

Which is consistent with \(P'(x) + f(x) = x\).

So \(y'' + xy' + y = 0\) can be written
\[
[1 \cdot y']' + [xy]' = 0
\]
Left side can be written as \([1 \cdot y' + xy]' = 0\) since derivative on left has value \(0\), \(y' + xy = C\) for some constant.

Now we have a solvable first order linear ODE.
3.2, #29) \[ y'' + 3x^2y' + xy = 0 \]
\[ p(x) = 1 \quad q(x) = 3x^2 \quad r(x) = x \]
\[ p'(x) = 0 \quad q'(x) = 6x \]
\[ p''(x) = 0 \]

\[ p''(x) - q'(x) + r(x) = 0 - 6x + x = -5x \neq 0. \quad \text{(Not a solution)} \]

3.2, #42) Find \( W(f, g) \) where \( f(\theta) = \cos 3\theta \), \( g(\theta) = 4 \cos^3 \theta - 3 \cos \theta \)

Function \( f \) involves cosine of multiple of \( \theta \) while \( g \) involves powers of \( \cos \theta \). It's probably helpful to rewrite \( f(\theta) = \cos 3\theta \) in terms of \( \sin \theta \) and \( \cos \theta \) for ease of manipulation. Think trig identities.

\[ \cos (3\theta) = \cos (2\theta + \theta) \]
\[ = [\cos 2\theta] \cos \theta - [\sin 2\theta] \sin \theta \]
\[ = [\cos^2 \theta - \sin^2 \theta] \cos \theta - [2 \sin \theta \cos \theta] \sin \theta \]
\[ = \cos^3 \theta - \sin^2 \theta \cos \theta - 2 \sin \theta \cos^2 \theta \]
\[ = \cos^3 \theta - 3 \sin \theta \cos^2 \theta \cos \theta \]
\[ = \cos^3 \theta - 3 \left[1 - \cos^2 \theta \right] \cos \theta \]
\[ = \cos^3 \theta - 3 \cos \theta + 3 \cos^3 \theta \]
\[ = 4 \cos^3 \theta - 3 \cos \theta \]

So we have \( f(\theta) = \cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta = g(\theta) \)
What does this tell us about \( W(f, g) \)?
3.11) If \( y_1, y_2 \) are linearly independent solutions to
\( y'' + p(t) y' + q(t) y = 0 \), prove that \( c_1 y_1 \) and \( c_2 y_2 \)
are also linearly independent solutions provided
neither \( c_1 \) nor \( c_2 \) is zero.

We know that \( c_1 y_1 \) and \( c_2 y_2 \) are solutions b/c
any multiple of a solution is also a solution.

To show linear independence, consider

\[
\begin{vmatrix}
  c_1 y_1 & c_2 y_2 \\
  c_1 y_1' & c_2 y_2'
\end{vmatrix}
\]

\[
= c_1 c_2 y_1 y_2' - c_1 c_2 y_1 y_2'
= c_1 c_2 (y_2 y_1' - y_1 y_2')
\]

Since \( c_1 \neq 0, c_2 \neq 0 \) by hypothesis, the Wronskian \( \neq 0 \)
provided \( y_1 y_2' - y_1' y_2 \neq 0 \). This is true b/c \( y_1, y_2 \)
are LINEARLY INDEPENDENT. So \( W(c_1 y_1, c_2 y_2) \neq 0 \).

So \( c_1 y_1 \) and \( c_2 y_2 \) are LINEARLY INDEPENDENT.

(Refer to Thm 3.3.3)

3.13) Suppose \( y_3, y_4 \) are linearly independent solutions to
\( y'' - p(t) y' + q(t) y = 0 \). Determine conditions such that
\( y_3 = a_1 y_1 + a_2 y_2 \) and \( y_4 = b_1 y_1 + b_2 y_2 \) are also lin. indep.

Look at \( W(y_3, y_4) \) and determine conditions on
\( a_1, a_2, b_1, b_2 \) that make \( W(y_3, y_4) \neq 0 \).
\[ W(y_3, y_4) = \begin{vmatrix}
  a_1 y_1 + a_2 y_2 & b_1 y_1 + b_2 y_2 \\
  a_1 y'_1 + a_2 y'_2 & b_1 y'_1 + b_2 y'_2
\end{vmatrix} \]

\[ = (a_1 y_1 + a_2 y_2)(b_1 y'_1 + b_2 y'_2) - (a_1 y'_1 + a_2 y'_2)(b_1 y_1 + b_2 y_2) \]

Multiply carefully — you get eventually 

\[ a_1 b_2 y_1 y_2 + a_2 b_1 y'_1 y'_2 - a_1 b_2 y'_1 y_2 - a_2 b_1 y_1 y'_2 \]

\[ = a_1 b_2 (y_1 y'_2 - y'_1 y_2) - a_2 b_1 (y_1 y'_2 - y'_1 y_2) \]

\[ = (a_1 b_2 - a_2 b_1)(y_1 y'_2 - y'_1 y_2) \]

We know \( \neq 0 \) b/c \( y_1, y_2 \) are linearly independent.

\[ W(y_3, y_4) \neq 0 \] as long as \( a_1 b_2 - a_2 b_1 \neq 0 \).
3. Suppose that if $y_1, y_2$ are zero at the same point in some interval $I$, then they cannot be fundamental set of solutions on that interval.

Suppose $y_1, y_2$ are two solutions of $y'' + p(t)y' + q(t)y = 0$ on $I$ and for some $a \in I$, $y_1(a) = y_2(a) = 0$.

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

$$= y_1(t)y_2'(t) - y_1'(t)y_2(t).$$

So $W(y_1, y_2)(a) = y_1(a)y_2'(a) - y_1'(a)y_2(a)$

$$= 0 \cdot y_2'(a) - y_1'(a) \cdot 0$$

$$= 0.$$ 

If $a \in I$ we have $W(y_1, y_2)(a) = 0$ so $W(y_1, y_2)$ is not different from 0 throughout interval $I$. Therefore $y_1, y_2$ are not linearly independent throughout $I$ and consequently cannot be fundamental solutions.

#27 and #28 follow similar line of argumentation.