(1) A random sample of 51 items resulted in a sample mean of 32.0 and a sample standard deviation of 6.0:

(a) Construct a 90% confidence interval for the population mean $\mu$.

It is given that $\bar{x} = 32.0$, $s = 6.0$, and $n = 51$

$1 - \alpha = 0.9 \implies \alpha = 0.1$

(the margin of error:) $E = t_{\frac{\alpha}{2},(n-1)}\left(\frac{s}{\sqrt{n}}\right) = t_{0.05, (51-1)} \left(\frac{6.0}{\sqrt{51}}\right) = t_{0.05, 50} \left(\frac{6.0}{\sqrt{51}}\right) = 1.676 \left(\frac{6.0}{\sqrt{51}}\right) = 1.408$

a 90% confidence interval for $\mu$ is $32.0 \pm 1.408 \implies (30.6, 33.4)$

It means: with 90% confidence, we claim that the interval (30.6, 33.4) contains the true value of the population mean $\mu$.

Note: The population standard deviation $\sigma$ is unknown, the sample standard deviation $s$ is used.

$\implies t$ probability distribution is used to find the critical value.

(b) Construct a 95% confidence interval for the population mean $\mu$.

$1 - \alpha = 0.95 \implies \alpha = 0.05$

(the margin of error:) $E = t_{\frac{\alpha}{2},(n-1)}\left(\frac{s}{\sqrt{n}}\right) = t_{0.025, (51-1)} \left(\frac{6.0}{\sqrt{51}}\right) = t_{0.025, 50} \left(\frac{6.0}{\sqrt{51}}\right) = 2.009 \left(\frac{6.0}{\sqrt{51}}\right) = 1.688$

a 95% confidence interval for $\mu$ is $32.0 \pm 1.688 \implies (30.3, 33.7)$

(c) Construct a 99% confidence interval for the population mean $\mu$.

$1 - \alpha = 0.99 \implies \alpha = 0.01$

(the margin of error:) $E = t_{\frac{\alpha}{2},(n-1)}\left(\frac{s}{\sqrt{n}}\right) = t_{0.005, (51-1)} \left(\frac{6.0}{\sqrt{51}}\right) = t_{0.005, 50} \left(\frac{6.0}{\sqrt{51}}\right) = 2.678 \left(\frac{6.0}{\sqrt{51}}\right) = 2.250$

a 99% confidence interval for $\mu$ is $32.0 \pm 2.25 \implies (29.8, 34.3)$

(d) Observe the interval widths in (a), (b), and (c), as the confidence level $(1 - \alpha)$ gets larger, how does the width of the confidence interval change (when using a same sample)?

In (a), the width of the 90% CI is: $33.4 - 30.6 = 2.8$
In (b), the width of the 95% CI is: $33.7 - 30.3 = 3.4$
In (c), the width of the 99% CI is: $34.3 - 29.8 = 4.5$

With the same sample, in order to raise the confidence level, the CI has to be enlarged.

(e) Suppose the sample size is 100 (instead of 51), what is the 95% confidence interval for $\mu$?

It is given that $\bar{x} = 32.0$, $s = 6.0$, and $n = 100$

$1 - \alpha = 0.95 \implies \alpha = 0.05$

$\implies E = t_{\frac{\alpha}{2},(n-1)}\left(\frac{s}{\sqrt{n}}\right) = t_{0.05, (100-1)} \left(\frac{6.0}{\sqrt{100}}\right) = t_{0.05, 99} \left(\frac{6.0}{\sqrt{100}}\right) = 1.984 \left(\frac{6.0}{\sqrt{100}}\right) = 1.190$

a 95% confidence interval for $\mu$ is $32.0 \pm 1.190 \implies (30.8, 33.2)$
(f) Compare (b) and (e), what is the relationship between the width of confidence interval and the sample size \( n \) (when \( n \) gets greater, how does the width of CI change?)

In (b), the 95\% CI for \( \mu \) is (30.3, 33.7) \( \implies \) (the width is 3.4) \( n = 51 \)

In (e), the 95\% CI for \( \mu \) is (30.8, 33.2) \( \implies \) (the width is 2.4) \( n = 100 \)

This fact tells us, with larger sample size, the CI becomes narrower. (The narrower the CI, the better.)

(2) Determine the appropriate critical value to construct a confidence interval for the population mean \( \mu \): (find the critical value, if neither of \( z \) nor \( t \) score will apply, write n/a).

(Case 1) The confidence level is 95\%, the sample size is \( n = 200 \), and \( \sigma = 9 \)

\( \implies \) When the population standard deviation \( \sigma \) is given, \( z \) probability distribution is used for the critical value. \( \implies z_{\frac{\alpha}{2}} = z_{0.025} = z_{0.025} = 1.96 \)

(Case 2) The confidence level is 95\%, the sample size is \( n = 20 \), and no information of the population Small sample and without population distribution information,

\( \implies \) the formula for CI will not apply.

(Case 3) The confidence level is 99\%, the sample size is \( n = 56 \), and no information of the population is given.

Although the population distribution is unknown, the sample is large, the formula for CI applies.

Since the population standard deviation \( \sigma \) is not given, \( t \) probability distribution is used for the critical value. \( \implies t_{\frac{\alpha}{2},(n-1)} = t_{0.025, (56-1)} = t_{0.005, 55} = 2.668 \)

(Case 4) The confidence level is 90\%, the sample size is \( n = 9 \), and the original population is normally distributed with \( \sigma = 14 \)

This is a small sample, but since the parent population is normally distributed, the formula for CI still applies.

Since the population standard deviation \( \sigma \) is given, \( z \) probability distribution is used for the critical value. \( \implies z_{\frac{\alpha}{2}} = z_{0.05} = 1.645 \)

(Case 5) The confidence level is 95\%, the sample size is \( n = 18 \), and the original population is normally distributed.

This is a small sample, but since the parent population is normally distributed, the formula for CI applies.

Since the population standard deviation \( \sigma \) is not given, \( t \) probability distribution is used for the critical value. \( \implies t_{\frac{\alpha}{2},(n-1)} = t_{0.05, (18-1)} = t_{0.025, 17} = 2.110 \)
(3) A random sample of 51 college graduates’ starting salary records resulted in a sample mean of $45678 and the sample standard deviation is $9900. Construct a 95% confidence interval to estimate the population standard deviation $\sigma$. (assume that the population has a normal distribution.)

\[ 1 - \alpha = 0.95 \implies \alpha = 0.05, \ n = 51, \ s = 9900 \quad (\bar{x} \text{ will not be used.}) \]

To estimate population standard deviation $\sigma$, $\chi^2$ probability distribution is used.

the lower limit for $\sigma^2$ is:
\[
\frac{(n-1) \ s^2}{\chi^2_{\alpha/2}, \ (n-1)} = \frac{(51-1) \ (9900)^2}{\chi^2_{0.025}, \ (51-1)} = \frac{(50)(9900)^2}{71.420} \approx 68615234
\]

the upper limit for $\sigma^2$ is:
\[
\frac{(n-1) \ s^2}{\chi^1_{\alpha/2}, \ (n-1)} = \frac{(51-1) \ (9900)^2}{\chi^1_{1-0.05}, \ (51-1)} = \frac{(50)(9900)^2}{32.357} \approx 151451000
\]

A 95% confidence interval for $\sigma$ is:
\[
\sqrt{68615234} < \sigma < \sqrt{151451000} \implies \$8283 < \sigma < \$12306
\]

(4) Scores on a certain test are normally distributed with a variance of 56. A researcher wishes to estimate the mean score achieved by all adults on the test. Find the sample size $n$ needed to assure with 98% confidence that the sample mean will not differ from the true population mean by more than 3 units.

the information / requirements are:

\[ \sigma^2 = 56 \quad \text{and} \quad 1 - \alpha = 0.98 \quad \text{and} \quad E = 3 \]

\[
\implies \text{(the sample size:)} \quad n = \left( \frac{z_{0.025}/\sqrt{\sigma}}{E} \right)^2 = \left( \frac{2.33/\sqrt{56}}{3} \right)^2 = \left( \frac{2.33}{3} \right)^2 \approx 33.78 = 34 \quad \text{(round up)}
\]

\[ \implies \text{the sample size should be at least} \quad 34. \]

(5) The football coach randomly selected ten players and timed how long each player took to perform a certain drill. The times (in minutes) were:

13.2, 11.3, 14.3, 9.7, 6.5, 11.6, 6.2, 6.6, 13.5, 10.2

Assume that the times are normally distributed. Construct a 95% confidence interval for the mean time for all players.

First, calculate to find: $\bar{x} = 10.31$, $s = 3.03$, and $n = 10$,

This sample is small, but since the parent population has a normal distribution, the formula for CI applies. ($\alpha = 0.05$)

Then, find the margin of error:

\[
\implies E = t_{\alpha/2, \ (n-1)} \left( \frac{s}{\sqrt{n}} \right) = t_{0.025, \ (10-1)} \left( \frac{3.03}{\sqrt{10}} \right) = t_{0.025, \ (9)} \left( \frac{3.03}{\sqrt{10}} \right) = 2.262 \left( \frac{3.03}{\sqrt{10}} \right) = 2.17
\]

a 95% CI (confidence interval) is: $\bar{x} \pm E = 10.31 \pm 2.17 \implies (8.14, 12.18) \text{ (minutes)}$

(6) A random sample is used to estimate the mean time required for caffeine from products such as coffee or soft drinks to leave the body after consumption. A 95% confidence interval based on this sample is: 5.6 hours to 6.4 hours (for adults).

(a) What is the sample mean?

Since the $1 - \alpha$ CI is constructed as $\bar{x} \pm E$, $\implies \bar{x}$ is the midpoint of the interval and $E$ is the radius.

\[ \implies \bar{x} = \frac{5.6+6.4}{2} = 6 \quad \text{(and} \quad E = \frac{6.4-5.6}{2} = 0.4) \]

(b) If the population standard deviation is known to be 2 hours. What is the sample size? ($n = ?$)

$\sigma = 2, \ \alpha = 0.05, \ \text{and} \ E = 0.4$ (see above)

The sample size is:

\[ n = \left( \frac{2 \alpha}{E} \right)^2 = \left( \frac{1.96}{0.4} \right)^2 = 96.04 \implies \text{the sample size is} \ n = 96 \text{ (or 97)} \]
(7) A survey conducted by the U.S. department of Labor found the 48 out of 500 heads of households were unemployed. Develop a 98% confidence interval estimate of the proportion of unemployed heads of households in the population.

the sample proportion is: \( \hat{p} = \frac{x}{n} = \frac{48}{500} = 0.096 \)

\( 1 - \alpha = 0.98 \implies \alpha = 0.02 \)

find the margin of error:

\[ E = z_{\frac{\alpha}{2}} \sqrt{\frac{(\hat{p})(1-\hat{p})}{n}} = z_{0.01} \sqrt{\frac{(0.096)(0.904)}{500}} = (2.33) \sqrt{\frac{(0.096)(0.904)}{500}} = 0.0307 \]

a 98% confidence interval for the population proportion \( p \) is:

\[ \hat{p} \pm E = 0.096 \pm 0.0307 = (0.065, 0.127) \text{ or (6.5%, 12.7%) } \]

(8) (Population normal distribution is assumed). The following numbers are waiting times (in minutes) of customers at a certain bank, where customers enter a single waiting line that feeds 3 teller windows. Construct a 95% confidence interval for the population standard deviation \( \sigma \).

6.5, 6.6, 6.7, 6.8, 7.1, 7.3, 7.4, 7.7, 7.7, 7.7

First, calculate the sample statistics: \( \implies \bar{x} = 7.15, \ x = 0.477, \text{ and } n = 10 \)

To construct a 95% confidence interval for \( \sigma \):

\[ \frac{(n-1)s^2}{\chi^2_{n-1, 0.025}} < \sigma < \frac{(n-1)s^2}{\chi^2_{n-1, 0.975}} \]

\[ \frac{(9)(0.477)^2}{19.023} < \sigma < \frac{(9)(0.477)^2}{2.700} \implies \sqrt{0.1076} < \sigma < \sqrt{0.7584} \implies 0.328 < \sigma < 0.870 \]