(1) Let a random variable $x$ be **Uniformly** distributed in the interval $[1, 15]$.

(a) What is the density function of this random variable?

$$f(x) = \begin{cases} \frac{1}{15-1} = \frac{1}{14} & \text{for } x \in [1, 15] \\ 0 & \text{elsewhere} \end{cases}$$

(b) $P(x > 5.5) = P(5.5 < x < 15) = \frac{15-5.5}{15-1} = \frac{9.5}{14} \approx 0.679$

(c) $P(3 < x < 7) = \frac{7-3}{15-1} = \frac{2}{7}$ (or $\approx 0.286$)

(2) $z$ is a standard normal random variable. Answer the following questions, together with a rough graph to show the corresponding area:

(a) $P(-0.95 \leq z \leq 1.59) = (\text{the area in between } z = -0.95 \text{ and } z = 1.59)$

$$= 0.9441 - 0.1711 = 0.7730$$

(b) $P(z \leq 0.72) = (\text{the area to the LEFT of } z = 0.72) = 0.7642$

(c) $P(-1.28 \leq z) = (\text{the area to the RIGHT of } z = -1.28) = 1 - 0.1003 = 0.8997$

(d) $P(z \geq 1.25) = (\text{the area to the RIGHT of } z = 1.25) = 1 - 0.8944 = 0.1056$

(e) If $P(z \geq a) = 0.591$; then $a \approx ?$

$$P(z \geq a) = 0.591 \implies P(z \leq a) = 1 - 0.591 = 0.409 \quad \text{(Note: find the LEFT area first.)}$$

From the body of Table A-2, we find $P(z \leq -0.23) = 0.4090 \implies a = -0.23$

(f) If $P(z \leq b) = 0.975$; then $b \approx ?$

From the body of Table A-2, we find $P(z \leq 1.96) = 0.9750 \implies b = 1.96$

(3) The random variable $x$ has a normal distribution,

(a) $x \sim N(60.3, 4.0)$, then, $P(x < 53.1) = P\left( \frac{x-60.3}{4.0} < \frac{53.1-60.3}{4.0} \right) = P(z < -1.8) = 0.0359$

(b) $x \sim N(137.0, 5.3)$, then, $P(134.4 < x < 140.1)$

$$= P\left( \frac{134.4-137.0}{5.3} < \frac{x-137.0}{5.3} < \frac{140.1-137.0}{5.3} \right) = P(-0.49 < z < 0.58) = 0.7190 - 0.3121 = 0.4069$$

(c) $x \sim N(13.8, 0.6)$, then, $0.242 = P(x > a) = P\left( \frac{x-13.8}{0.6} > \frac{a-13.8}{0.6} \right) = P(z > \frac{a-13.8}{0.6}) = 0.242$, From Table A-2, $P(z > 0.70) = 0.242 \implies \frac{a-13.8}{0.6} = 0.70$

( because $P(z < 0.70) = 0.7580 \implies P(z > 0.70) = 1 - 0.7580 = 0.2420$ )

Solve the above linear equation

$$\implies a - 13.8 = 0.6 \times 0.70 \implies a - 13.8 = 0.42 \implies a = 13.8 + 0.42 = 14.22$$

(4) The time required to complete a final exam in a particular college course is normally distributed, with a mean of 80 minutes and a standard deviation of 12 minutes. Answer the following questions:
(a) (for a randomly selected student) What is the probability of completing the exam in 1 hour or less?

Let \( x \) be the time needed for a student to complete the exam, it is given in the problem that \( x \sim N(80, 12) \) (Note: the unit is minute.)

\[
P(x < 60) = P(z < \frac{60-80}{12}) = P(z < -1.67) = 0.0475 \text{ (or 4.75\%)}
\]

That means: the chance is very small for a student can complete the exam in 60 minutes. Or, among 100 students, only about 5 students (4.75\%) can complete the exam in 1 hour.

(b) What is the probability a randomly selected student will complete the exam in more than 60 minutes but less than 75 minutes?

\[
P(60 < x < 75) = P\left(\frac{60-80}{12} < z < \frac{75-80}{12}\right) = P(-1.67 < z < -0.42) = 0.3372 - 0.0475 = 0.2897 \text{ (or 28.97\%)}
\]

(c) Assume that the class has 45 students and that the exam period is 90 minutes in length. How many students do you expect will be unable to complete the exam in the allotted time? (round it to a whole number.) Hint: find the probability as a percentage, and then multiply the total number of students by the percentage.

"UNable to complete the exam in 90 minutes" means the time needed is more than 90 minutes.

\[
P(x > 90) = P(z > \frac{90-80}{12}) = P(z > 0.83) = 1 - P(z < 0.83) = 1 - 0.7967 = 0.2033 \text{ (or 20.33\%)}
\]

If there are 45 students in the class, there will be about \( 45 \times 0.2033 \approx 9 \) students who cannot complete the exam in 90 minutes.

(5) Assume that men’s weights are normally distributed with \( \mu = 172 \text{ lb} \) and \( \sigma = 29 \text{ lb} \).

(a) If one man is randomly selected, find the probability that his weight is greater than 180 lb.

Let \( x \) be the man’s body weight, \( \implies x \sim N(172, 29) \)

\[
P(x > 180) = P(z > \frac{180-172}{29}) = P(z > 0.28) = 1 - P(z < 0.28) = 1 - 0.6103 = 0.3897 \text{ (or 38.97\%)}
\]

that means: if an adult man is selected at random, the chance is a little less than 40\% for you to find that his weight is over 180 lb. (or, among 100 randomly selected men, about 39 of them are with body weight over 180 lb.)

(b) If 100 men are randomly selected, find the probability that their mean weight is greater than 180 lb.

Let \( \bar{x} \) be the mean (average) body weight of the random sample of \( n = 100 \).

It is a large sample, according to CLT, \( \implies \bar{x} \sim N(172, \frac{29}{\sqrt{100}}) = N(172, 2.9) \)

Then, \( P(\bar{x} > 180) = P(z > \frac{180-172}{2.9}) = P(z > 2.76) = 1 - P(z < 2.76) = 1 - 0.9971 \)

\[
= 0.0029 \text{ (or 0.29\%)}
\]

that means: it is very very unlikely that you will find the average body weight of 100 randomly selected men to be over 180 lb.
(6) The weights of the fish in a certain pond are normally distributed with a mean of 15 lb and a standard deviation of 6 lb. If 4 fish are randomly selected, what is the probability that the mean weight will be between 12.6 and 18.6 lb?

Let \( \bar{x} \) be the mean (average) fish weight of the random sample of \( n = 4 \),

It is a **small** sample, but since its parent population has a normal distribution, CLT still applies.

\[ \Rightarrow \bar{x} \sim N(15, \frac{6}{\sqrt{4}}) = N(15, 3) \]

Then,

\[ P(12.6 < \bar{x} < 18.6) = P\left(\frac{12.6 - 15}{\frac{6}{\sqrt{4}}} < z < \frac{18.6 - 15}{\frac{6}{\sqrt{4}}}\right) = P(-0.80 < z < 1.20) = 0.8849 - 0.2119 = 0.6730 \]

(7) Answer the following questions:

(a) From a normally distributed population, samples of size \( n = 8 \) are randomly selected. Can we assume that the sample means are normally distributed?

**Yes.** Sample mean (\( \bar{x} \)) will have a normal distribution as long as its parent population has a normal distribution, no matter whether the sample size (\( n \)) is large or small.

(b) From a population with arbitrary distribution, samples of size \( n = 8 \) are randomly selected. Can we assume that the sample means are normally distributed?

**No.** (It’s a small sample with parent population distribution unknown.)

(c) From a population with unknown distribution, random samples are selected. What requirement is needed then we can assume that the sample means are normally distributed?

**Large** sample. \( (n > 30) \).

(8) In planning for hot water requirements, the manager of the Luxurion Hotel finds that guests spend a mean of 11.4 min each day in the shower. Assume that the shower times are normally distributed with a standard deviation of 2.6 min.

(a) Find the percentage of guests who shower more than 12 min. (i.e. the probability for a randomly selected guest to shower more than 12 min.)

Let \( x \) be the guest show time (in minutes), \( \Rightarrow x \sim N(11.4, 2.6) \)

\[ P(x > 12) = P\left(z > \frac{12 - 11.6}{2.6}\right) = P(z > 0.23) = 1 - P(z < 0.23) = 1 - 0.5910 = 0.4090 \text{ (or 40.9%)} \]

that means: For an individual guest, the chance to have a show time more than 12 minutes is not unusual. (about 40%)

(b) The hotel has installed a system that can provide enough hot water provided that the mean shower time for 84 guests is less than 12 min. If the hotel currently has 84 guests, find the probability that there will not be enough hot water.

Let \( \bar{x} \) be the mean (average) show time of the random sample of \( n = 84 \),

It is a **large** sample, according to CLT, \( \Rightarrow \bar{x} \sim N(11.4, \frac{2.6}{\sqrt{84}}) = N(11.4, 0.2837) \)

"there will not be enough hot water" means \( \bar{x} \geq 12 \). (because if the average shower time is less than 12 minutes, the hot water supply is enough.)

\[ P(\bar{x} > 12) = P\left(z > \frac{12 - 11.4}{\frac{2.6}{\sqrt{84}}}\right) = P(z > 2.11) = 1 - P(z < 2.11) = 1 - 0.9826 = 0.0174 \text{ (or 1.74%)} \]

that means: Chance is very very small that the hot water supply will be not enough for 84 guests. It is almost impossible for the average shower time of 84 guests to be more than 12 minutes.