Matrices Light A La Green

Matrix Row Operations:
These row operations produce matrices that represent systems with the same solutions.

1) Two rows of a matrix may be interchanged. This is the same as interchanging two equations in a linear system.

2) The elements in any row may be multiplied by a nonzero number. This is the same as multiplying both sides of an equation by a nonzero number. I tend to divide the elements in a row but that is cool because multiplication and division are inverse operations.

3) The elements in any row may be multiplied by a nonzero number, and these products may be added to the corresponding elements in any other row. This is the same as multiplying both sides of an equation by a nonzero number and then adding equations to eliminate a variable.

Example 1. Two equations with two variables:
\[ x + y = 3 \]
\[ x - y = -1 \]

Write the augmented matrix for the system.

\[
\begin{bmatrix}
1 & 1 & 3 \\
1 & -1 & -1 \\
\end{bmatrix}
\]

We are going to use our row operations to get the matrix in the following form:

\[
\begin{bmatrix}
1 & a & b \\
0 & 1 & c \\
\end{bmatrix}
\]

Notice we want a 1 in the top row, 1st element, a 0 in the second row, 1st element, and a 1 in the second row, second element.

\[
\begin{bmatrix}
1 & 1 & 3 \\
1 & -1 & -1 \\
\end{bmatrix}
\]
We already have a 1 in the 1st row, 1st element. So we now multiply $-1$ times the elements in row 1 plus the elements in row 2 to get a 0 in the second row, 1st element.

$$-1 \cdot R_1 + R_2 = New R_2$$

\[
\begin{bmatrix}
1 & 1 & 3 \\
0 & -2 & -4
\end{bmatrix}
\]

We now need a 1 in the second row, 1st element, so we divide each of the elements in row 2 by $-2$.

$$\frac{R_2}{-2} = New R_2$$

\[
\begin{bmatrix}
1 & 1 & 3 \\
0 & 1 & 2
\end{bmatrix}
\]

We now reinsert the variables to find the solution to the system.

\[
x + y = 3 \\
y = 2
\]

\[
x + 2 = 3 \\
x = 1
\]

The solution to the system is $(1, 2)$. ★

Example 2. Three equations with three variables:

\[
3x + y + 2z = 31 \\
x + y + 2z = 19 \\
x + 3y + 2z = 25
\]

Write the augmented matrix for the system.

\[
\begin{bmatrix}
3 & 1 & 2 & 31 \\
1 & 1 & 2 & 19 \\
1 & 3 & 2 & 25
\end{bmatrix}
\]
We are going to use our row operations to get the matrix in the following form.

\[
\begin{bmatrix}
1 & a & b & c \\
0 & 1 & d & e \\
0 & 0 & 1 & f
\end{bmatrix}
\]

Exchange rows 1 and 2 so that we have a 1 in the first row, first element.

\[
\begin{bmatrix}
1 & 1 & 2 & 19 \\
3 & 1 & 2 & 31 \\
1 & 3 & 2 & 25
\end{bmatrix}
\]

Multiply the elements in row 1 by $-3$ and add them to the elements in row 2 in order to have a 0 in the second row, 1st element.

$-3 \cdot R_1 + R_2 = New R_2$

\[
\begin{bmatrix}
1 & 1 & 2 & 19 \\
0 & -2 & -4 & -26 \\
1 & 3 & 2 & 25
\end{bmatrix}
\]

Divide each of the elements in row 2 by $-2$ in order to have a 1 in the second row, second element.

\[
\begin{bmatrix}
1 & 1 & 2 & 19 \\
0 & 1 & 2 & 13 \\
1 & 3 & 2 & 25
\end{bmatrix}
\]

Multiply the elements in row 1 by $-1$ and then add them to the elements in row 3 in order to have a 0 in the third row, 1st element.

$-1 \cdot R_1 + R_3 = New R_3$

\[
\begin{bmatrix}
1 & 1 & 2 & 19 \\
0 & 1 & 2 & 13 \\
0 & 2 & 0 & 6
\end{bmatrix}
\]
Multiply the elements in row 2 by $-2$ and then add them to the elements in row 3 in order to have a 0 in the third row, 1st element.

$$-2 \cdot R_2 + R_3 = New \ R_3$$

\[
\begin{bmatrix}
1 & 1 & 2 & 19 \\
0 & 1 & 2 & 13 \\
0 & 0 & -4 & -20
\end{bmatrix}
\]

Divide the elements in row 3 by $-4$ to have a 1 in the third row, 1st element.

$$\frac{R_3}{-4} = New \ R_3$$

\[
\begin{bmatrix}
1 & 1 & 2 & 19 \\
0 & 1 & 2 & 13 \\
0 & 0 & 1 & 5
\end{bmatrix}
\]

We now reinsert the variables to find the solution of system.

\[
x + y + 2z = 19
\]

\[
y + 2z = 13
\]

\[
z = 5
\]

Back-substitute into equation #2.

\[
y + 2 \cdot 5 = 13
\]

\[
y + 10 = 13
\]

\[
y = 3
\]

Back-substitute into equation #1.

\[
x + y + 2z = 19
\]

\[
x + 3 + 2 \cdot 5 = 19
\]

\[
x + 13 = 19
\]

\[
x = 6
\]

The solution of the system is $(6, 3, 5)$. ★

We can then check the solution in equations 2 and 3.

\[
x + y + 2z = 19 \quad \rightarrow \quad 6 + 3 + 2 \cdot 5 = 19 \checkmark
\]

\[
x + 3y + 2z = 25 \quad \rightarrow \quad 6 + 3 \cdot 3 + 2 \cdot 5 = 25 \checkmark
\]