Inverses of Functions

I) A set of ordered pairs is called a relation. When we consider the graph of a function, we are thinking about a set of ordered pairs. Thus, a function can be thought of as a special kind of relation, in which no \( x\)-coordinate can have more than one corresponding \( y\)-coordinate.

Let us start with the relation \( A = \{(1, 5), (2, 10), (3, 15), (4, 20)\}\). If we interchange the \( x\)-coordinates and \( y\)-coordinates, the relation we obtain is called the inverse of the relation \( A \). The inverse of the relation \( A \) is \( \{(5, 1), (10, 2), (15, 3), (20, 4)\}\). The inverse is a function because no \( x\)-coordinate has more than one corresponding \( y\)-coordinate.

Problem 1: Find the inverse of the relation \( B = \{(-2, \frac{1}{4}), (-1, \frac{1}{2}), (0, 1), (1, 2), (2, 4)\}\). Tell whether the inverse is a function.

Solution: The inverse of the relation \( B \) is \( \{(-\frac{1}{4}, -2), (-\frac{1}{2}, -1), (1, 0)\}\). It is a function because no \( x\)-coordinate has more than one corresponding \( y\)-coordinate.

II) We know that a function can be represented by an equation. We know that \( f(x) = 4x - 3 \) or \( y = 4x - 3 \) is a linear function. We can find input and corresponding output values by using a table of values.

\[
\begin{array}{c|c|c|c}
 x & 0 & 1 & 2 \\
 y & -3 & 1 & 5 \\
\end{array}
\]

Note that whatever value we use for \( x \), we will get a different \( y\)-value no matter what values we use for \( x \). No \( y\)-values will repeat. When this happens, we call the function one-to-one, because each input value of \( x \) in the domain determines a different output value of \( y \) in the range.

Problem 2: Determine whether the function \( y = x^2 + 2 \) is a one-to-one function by using a table of values.

Solution: 

\[
\begin{array}{c|c|c|c|c|c}
 x & -2 & -1 & 0 & 1 & 2 \\
 y & 6 & 3 & 2 & 3 & 6 \\
\end{array}
\]

The function \( y = x^2 + 2 \) is not a one-to-one function because the inputs \(-2\) and \(2\) both have the same output \(6\). Likewise, the inputs \(-1\) and \(1\) both have the same output \(3\).
Problem 3: Determine whether the function \( y = x^3 \) is a \textit{one-to-one function} by using a table of values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-2)</th>
<th>(-1)</th>
<th>(0)</th>
<th>(1)</th>
<th>(2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>(-8)</td>
<td>(-1)</td>
<td>(0)</td>
<td>(1)</td>
<td>(8)</td>
</tr>
</tbody>
</table>

The function \( y = x^3 \) is a \textit{one-to-one function} because whatever value we might choose for \( x \), we would never have the output values for \( y \) repeat. ■

III) Another way to tell if a function is a \textit{one-to-one function} is to look at its graph. We can draw horizontal lines to decide whether the graph represents a \textit{one-to-one function}. If every horizontal line that we draw intersects the graph of a function at only one point, then we know the graph represents a \textit{one-to-one function}. If any horizontal line intersects the graph two or more times, then the graph does not represent a \textit{one-to-one function}!

Problem 4: Use the horizontal line test to determine whether the graph of the function \( y = x^2 + 2 \) represents a \textit{one-to-one function}.

Because we can draw at least one horizontal line that intersects the graph more than once, the graph of \( y = x^2 + 2 \) does not represent a \textit{one-to-one function}. ■
Problem 5: Use the horizontal line test to determine whether the graph of the function \( y = x^3 \) represents a one-to-one function.

Because every horizontal line we draw intersects the graph only once, the graph of \( y = x^3 \) does represent a one-to-one function.

IV) Why are one-to-one functions so special? I mean, what is the big deal? They are significant because when a function is one-to-one, its inverse is a function as well! On the flip side, if a function is not one-to-one, then its inverse is not a function!!!

If we have one-to-one function, its inverse is a function and is found using the following steps.

A) Replace \( f(x) \) with \( y \) if necessary.
B) Interchange the variables \( x \) and \( y \).
C) Solve the resulting equation for \( y \).
D) Replace \( y \) with the notation \( f^{-1}(x) \) to indicate that the new equation is the inverse of \( f(x) \).

Problem 6: Given the one-to-one function \( f(x) = 4x - 3 \), find its inverse.

Solution:
A) \( y = 4x - 3 \)
B) \( x = 4y - 3 \)
C) \(4y - 3 = x\) (Switch the two sides of the equation.)

\[
4y = x + 3 \\
y = \frac{x + 3}{4} \text{ or } y = \frac{1}{4}x + \frac{3}{4}
\]

D) \(f^{-1}(x) = \frac{1}{4}x + \frac{3}{4}\)

We notice in the tables below what happens with the domain and range of the function and its inverse. The domain and the range interchanged! This can be very helpful because when we find the point pairs for the original function, we can interchage each of those point pairs to find the table of values for its inverse and then go on to graph both the function and its inverse.

\[
\begin{array}{c|c}
 x & y \\
\hline
0 & -3 \\
1 & 1 \\
2 & 5 \\
\end{array}
\]

\[
\begin{array}{c|c}
 x & y \\
\hline
-3 & 0 \\
1 & 1 \\
5 & 2 \\
\end{array}
\]

Problem 7: Given the one-to-one function \(f(x) = x^3\), find its inverse.

**Solution:**

A) \(y = x^3\)

B) \(x = y^3\)

C) \(y^3 = x\) (Switch the two sides of the equation.)

\[
y = \sqrt[3]{x}
\]

D) \(f^{-1}(x) = \sqrt[3]{x}\)

Looking at tables of values for the function \(y = x^3\) and its inverse \(y = \sqrt[3]{x}\), we notice again that the domain and range interchanged!

\[
\begin{array}{c|c}
 x & y \\
\hline
-2 & -8 \\
-1 & -1 \\
0 & 0 \\
1 & 1 \\
2 & 8 \\
\end{array}
\]

\[
\begin{array}{c|c}
 x & y \\
\hline
-8 & -2 \\
-1 & -1 \\
0 & 0 \\
1 & 1 \\
8 & 2 \\
\end{array}
\]
Problem 8: Find the inverse of the one-to-one function \( y = 5x + 2 \). Then graph the function and its inverse on one coordinate system. Include the line of symmetry \( y = x \) on your graph.

**Solution:**

A) \( y = 5x + 2 \)

B) \( x = 5y + 2 \)

C) \( 5y + 2 = x \) (Switch the two sides of the equation.)

\[
5y = x - 2
\]

\[
y = \frac{x - 2}{5}
\]

or \( y = \frac{1}{5}x - \frac{2}{5} \)

D) \( f^{-1}(x) = \frac{1}{5}x - \frac{2}{5} \)

\[
y = 5x + 2
\]

\[
\begin{array}{c|c}
 x & y \\
-1 & -3 \\
0 & 2 \\
1 & 7 \\
\end{array}
\]

\[
y = \frac{1}{5}x - \frac{2}{5}
\]

\[
\begin{array}{c|c}
 x & y \\
-3 & -1 \\
2 & 0 \\
7 & 1 \\
\end{array}
\]