COMPLEX NUMBERS

1) We define the number such that and .
   a) \(\sqrt{-25} = \sqrt{25 \cdot -1} = \sqrt{25} \cdot \sqrt{-1} = 5i\)
   b) \(\sqrt{-48} = \sqrt{48 \cdot -1} = \sqrt{4 \cdot 3^2 \cdot -1} = 2^2 \sqrt{3} \cdot i = 4 \sqrt{3} i\) (Note \(i\) is not under radical!)
   c) \(-\sqrt{-60} = -\sqrt{60 \cdot -1} = -\sqrt{2^2 \cdot 3^2 \cdot 5 \cdot -1} = -1 \cdot 2 \sqrt{15} i = -2 \sqrt{15} i\)

3) A complex number is any number that can be written in the form \(a + bi\) where \(a\) and \(b\) are real numbers. We can add, subtract, or multiply them as we do binomials.
   a) \((8 + 6i) + (3 + 2i) = 1(8 + 6i) + 1(3 + 2i) = 8 + 6i + 3 + 2i = 11 + 8i\)
   b) \((4 + 5i) - (6 - 3i) = 1(4 + 5i) - 1(6 - 3i) = 4 + 5i - 6 + 3i = -2 + 8i\)
   c) \((3 - \sqrt{-36}) - (-3 + \sqrt{-36}) = 1(3 - 6i) - 1(-3 + 6i) = 3 - 6i + 3 - 6i = 6 - 12i\)
   d) \(4(3 - 5i) = 12 - 20i\)
   e) \((1 + 2i)(1 + 3i) = 1 + 3i + 2i + 6i^2 = 1 + 3i + 2i + (-6) = -5 + 5i\)

5) The conjugate of a complex number \(a + b i\) is \(a - bi\), and the conjugate of \(a - bi\) is \(a + bi\).
   a) The conjugate of \(5 + 7i\) is \(5 - 7i\).
   b) The conjugate of \(14 - 3i\) is \(14 + 3i\).

6) When we multiply complex conjugates, we get a real number as the product.
   a) \((3 + 4i)(3 - 4i) = 9 - 16i^2 = 9 + 16 = 25\)
   b) \((5 - 2i)(5 + 2i) = 25 - 4i^2 = 25 + 4 = 29\)

7) Conjugates are used when dividing complex numbers. We multiply the numerator and the denominator by the conjugate of the denominator, just like we have been when we were rationalizing the denominator in past work. Note: Do not distribute in the numerator until the very last step if the numerator is a real number and not a complex number.

   a) \(\frac{26}{2 - 3i} \cdot \frac{2 + 3i}{2 + 3i} = \frac{26(2 + 3i)}{4 - 9i^2} = \frac{26(2 + 3i)}{4 + 9} = \frac{26(2 + 3i)}{13} = 2(2 + 3i) = 4 + 6i\)

   b) \(\frac{-181}{10 + 9i} \cdot \frac{10 - 9i}{10 - 9i} = \frac{-181(10 - 9i)}{100 - 81i^2} = \frac{-181(10 - 9i)}{100 + 81} = \frac{-181(10 - 9i)}{181} = -1(10 - 9i) = -10 + 9i\)

   c) \(\frac{-5 + 9i}{1 - 2i} \cdot \frac{1 + 2i}{1 + 2i} = \frac{-5 + 10i + 9i + 18i^2}{1 - 4i^2} = \frac{-5 - i - 18}{1 + 4} = \frac{-23 - i}{5}\)

   \(= \frac{-23}{5} \cdot \frac{-i}{5} = \frac{-23}{5} \cdot \frac{1}{5} i\)
8) Powers of $i$:

Given $i = \sqrt{-1}$

- $i^2 = -1$

$\therefore$ 

Given $i^2 = -1$

- $i^6 = i^4 \cdot i^2 = 1 \cdot (-1) = -1$

Then $i^3 = i^2 \cdot i = -1 \cdot i = -i$

- $i^7 = i^4 \cdot i^3 = 1 \cdot -i = -i$

Then $i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1$

- $i^8 = i^4 \cdot i^4 = 1 \cdot 1 = 1$

$a)$ $i^{27} = i^4 \cdot i^4 \cdot i^4 \cdot i^4 \cdot i^4 \cdot i^3 = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot i^3 = -i$

$b)$ or $i^{27} \div 4 = 6 R 3$, therefore $i^{27} = i^3 = -i$

9) We can use different powers of $i$ to simplify fractions that have imaginary denominators.

$a)$ $\frac{5}{i} \cdot i = \frac{5i}{i^2} = \frac{5i}{-1} = -5i = 0 - 5i$

$b)$ $\frac{-3}{i^3} \cdot i = \frac{-3i}{i^4} = \frac{-3i}{1} = -3i = 0 + 3i$

10) When we solve quadratic equations, our solutions or roots can be complex conjugates.

$a)$ $x^2 - 6x + 10 = 0$ solved by completing the square...

$x^2 - 6x + \boxed{9} = -10 + \boxed{9}$

$(x - 3)^2 = -1$

$\sqrt{(x - 3)^2} = \sqrt{-1}$

$|x - 3| = i$

- $x - 3 = i$ or $x - 3 = -i$

- $x = 3 + i$ or $x = 3 - i$

$a)$ $x^2 - 6x + 10 = 0$ solved using the quadratic formula...

$a = 1, b = -6, \text{ and } c = 10$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot 10}}{2 \cdot 1} = \frac{6 \pm \sqrt{36 + (-40)}}{2}$

$= \frac{6 \pm \sqrt{-4}}{2} = \frac{6 \pm 2i}{2} = \frac{3 \pm i}{1} = 3 \pm i = 3 + i \text{ or } 3 - i$
**Quadratic Functions**

1) A quadratic function is a second-degree polynomial function of the form 
   \[ y = f(x) = ax^2 + bx + c \ (a \neq 0) \] 
   where \( a, b, \) and \( c \) are real numbers. They can be rewritten in the form 
   \[ y = a(x - h)^2 + k. \]

2) The graph of the function \( y = a(x - h)^2 + k \ (a \neq 0) \) is a parabola with vertex at \((h, k)\).
   The parabola opens upward when \( a > 0 \) and downward when \( a < 0 \). The axis of symmetry is the line 
   \( x = h \). When the parabola opens upward, the vertex is a minimum value and when the parabola opens 
   downward, the vertex is a maximum value.

3) **Rewrite \( y = x^2 - 4x + 4 \) in the form \( y = a(x - h)^2 + k \) by completing the square and graph the parabola.**
   
   \[
   y = x^2 - 4x + 4 = (x - 2)^2 + 0
   \]
   
   *The vertex is \((2, 0)\).*

   *Choose 3 lesser values than 2 and 3 greater values than 2 for \( x \) to find more points on the parabola and then make a table of values.*

<table>
<thead>
<tr>
<th></th>
<th>(-1)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>9</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>9</td>
</tr>
</tbody>
</table>

Note that the vertical line \( x = 2 \) acts as the axis of symmetry or a center line for the graph of the parabola on the left.

Also, the point with coordinates \((2, 0)\) is the vertex and also a minimum value.
4) Rewrite \( y = 2x^2 - 20x \) in the form \( y = a(x - h)^2 + k \) and find the vertex and the axis of symmetry.

\[
\begin{align*}
y &= 2x^2 - 20x \\
y &= 2(x^2 - 10x) & \text{Factor only the numerical coefficient of } x^2. \\
y &= 2(x^2 - 10x + 
\frac{25}{2}) - 2 \cdot 
\frac{25}{2} & \text{Complete the square. We are adding 50 and subtracting 50.} \\
y &= 2(x - 5)^2 - 50 & \text{Simplify.}
\end{align*}
\]

The vertex is \((5, -50)\) and the axis of symmetry is \(x = 5\). ■

5) Rewrite \( y = 3x^2 - 18x + 17 \) in the form \( y = a(x - h)^2 + k \) form and find the vertex and axis of symmetry. Tell whether the vertex is a maximum or minimum value.

\[
\begin{align*}
y &= 3x^2 - 18x + 17 \\
y &= (3x^2 - 18x) + 17 & \text{Insert parentheses around the first two terms, } 3x^2 \text{ and } 18x. \\
y &= 3(x^2 - 6x) + 17 & \text{Factor out the leading coefficient of the first two terms, } 3. \\
y &= 3(x^2 - 6x + 
\frac{9}{1}) - 3 \cdot 
\frac{9}{1} + 17 & \text{Get ready to complete the square.} \\
y &= 3(x^2 - 6x + \frac{9}{1}) - 3 \cdot \frac{9}{1} + 17 & \text{Complete the square. We are adding 27 and subtracting 27.} \\
y &= 3(x - 3)^2 - 27 + 17 & \text{Factor the 1st three terms inside the parentheses and multiply } 3 \cdot 9. \\
y &= 3(x - 3)^2 - 10 & \text{Combine the constants as like terms.}
\end{align*}
\]

The vertex is \((3, -10)\) and the axis of symmetry is \(x = 3\). Because the parabola opens up, the vertex is a minimum value. ■

6) Rewrite \( y = -2x^2 - 5x + 3 \) in the form \( y = a(x - h)^2 + k \) form and find the vertex and axis of symmetry. Tell whether the vertex is a maximum or minimum value.

\[
\begin{align*}
y &= -2x^2 - 5x + 3 \\
y &= \left(2 \cdot \frac{5}{2} x\right) + 3 & \text{Factor the leading coefficient, } -2. \\
y &= \left(2x^2 + \frac{5}{2} x\right) + 3 & \text{Get ready to complete the square.} \\
y &= \left(2x^2 + \frac{5}{2} x + \frac{25}{16}\right) + 2 \cdot 
\frac{25}{16} + 3 & \frac{1}{2} \text{ of } \frac{5}{2} \text{ is } \frac{5}{4}. \left(\frac{5}{4}\right)^2 = \frac{25}{16}. \\
y &= \left(2x + \frac{5}{4}\right)^2 + \frac{25}{8} + 3 & \text{Factor the first three terms and multiply } 2 \cdot \frac{25}{16} = \frac{25}{8}. \\
y &= \left(2x + \frac{5}{4}\right)^2 + \frac{25}{8} + \frac{24}{8} & \text{Convert 3 to a fraction whose denominator is } 8. \frac{1}{1} = \frac{8}{8}. \\
y &= \left(2x + \frac{5}{4}\right)^2 + \frac{49}{8} & \text{Combine the constants; } \frac{25}{8} + \frac{24}{8} = \frac{49}{8}.
\end{align*}
\]

The vertex is \((-\frac{5}{4}, \frac{49}{8})\) and the axis of symmetry is \(x = -\frac{5}{4}\). Because the parabola opens down, the vertex is a maximum value. ■
The x-coordinate of the vertex point can be given by \( x = \frac{-b}{2a} \).
This fact can be essential as a check to determine that one has successfully found the vertex using the completing the square method.

Find the vertex of \( y = -2x^2 - 5x + 3 \) using the formula \( x = \frac{-b}{2a} \).

\[
\begin{align*}
x &= \frac{-(-5)}{2 \cdot (-2)} = \frac{5}{-4} = -\frac{5}{4} \\
\text{If } x &= -\frac{5}{4} \text{, then } y = -2 \cdot \left( -\frac{5}{4} \right)^2 - 5 \cdot \left( -\frac{5}{4} \right) + 3 = -2 \cdot \left( \frac{25}{16} \right) + \frac{25}{4} + 3 \\
&= -\frac{25}{8} + \frac{25}{4} + \frac{3}{1} = -\frac{25}{8} + \frac{50}{8} + \frac{24}{8} = \frac{49}{8}. \text{ The vertex is } \left( -\frac{5}{4}, \frac{49}{8} \right) \quad \blacksquare
\end{align*}
\]

8) A landscaper has 88 feet of fencing which can be used to enclose a rectangular garden. What dimensions for the garden will enclose the maximum amount of area?

1) Let \( x = \text{ the width of the rectangular garden } \) and let \( y = \text{ the area of the rectangle } \).

If the width is \( x \) and one width and one length is 44 feet or half way around the rectangle, then the length must be \( 44 - x \).

2) Then \( 44 - x = \text{ the length of the rectangular garden } \).

3) \( \text{area} = \text{width} \cdot \text{length} \)

\[ y = x \cdot (44 - x) \]

4) \( y = 44x - x^2 \)
\[
\begin{align*}
y &= -x^2 + 44x \\
y &= -1(x^2 - 44x) \\
y &= -1(x^2 - 44x + \square) + 1 \cdot \square \\
y &= -1(x^2 - 44x + \frac{484}{1}) + 1 \cdot \frac{484}{1} \\
y &= -1(x - 22)^2 + 484 \\
\end{align*}
\]

6) The vertex of the parabola is \((22, 484)\). Because the parabola opens downward, the vertex is a maximum value. Thus, the maximum value for the width is 22 feet and the maximum length is \( 44 - x = 44 - 22 = 22 \) feet. The maximum value for the area is 484 square feet. \quad \blacksquare
9) Graph \( y = \frac{1}{2} (x + 4)^2 - 3 \). The vertex is \((-4, -3)\) and the parabola opens upward. The value of \( \frac{1}{2} \) for \( a \) makes the parabola wider. (When \( 0 < a < 1 \), the parabola is relatively wider.) Choose three lesser values than 4 and three greater values than 4 for \( x \) to find more points on the parabola.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-7</td>
<td>1.5</td>
</tr>
<tr>
<td>-6</td>
<td>-1</td>
</tr>
<tr>
<td>-5</td>
<td>-2.5</td>
</tr>
<tr>
<td>-4</td>
<td>-3</td>
</tr>
<tr>
<td>-3</td>
<td>-2.5</td>
</tr>
<tr>
<td>-2</td>
<td>-1</td>
</tr>
<tr>
<td>-1</td>
<td>1.5</td>
</tr>
</tbody>
</table>

Note that the vertical line \( x = -4 \) acts as the axis of symmetry or a center line for the graph of the parabola.

Also, the point with coordinates \((-4, -3)\) is the lowest point on the graph and thus is a minimum value.