Solving Linear Systems of Equations

Case 1) Intersecting Lines: If the lines are different and intersect, the equations are independent and the system is consistent. ONE solution exists.

Example 1: Solve the system by graphing.

\[x - y = 4\]
\[2x + y = 5\]

Because the equations are in general form, we can graph the lines by finding the intercepts and an insurance point in a table of values. Or we can convert to slope–intercept form and graph the lines using the slope and y–intercept.

\[
\begin{array}{c|c}
\text{} & \text{x} - y = 4 \\
\hline
x & y \\
0 & -4 \\
4 & 0 \\
2 & -2 \\
\end{array}
\]

\[
\begin{array}{c|c}
\text{} & \text{2x + y = 5} \\
\hline
x & y \\
0 & 5 \\
5 & 0 \\
5 & -5 \\
\end{array}
\]

This system is consistent with one solution, \((3, -1)\).
The equations are different so they are independent equations.

To verify \((3, -1)\) is the solution to the system, check it both equations.

\[x - y = 4 \quad \rightarrow \quad 3 - (-1) = 4 \quad \rightarrow \quad 3 + 1 = 4 \checkmark\]
\[2x + y = 5 \quad \rightarrow \quad 2 \cdot (3) + (-1) = 5 \quad \rightarrow \quad 6 + (-1) = 5 \checkmark\]
Case 2) Parallel Lines: If the lines are different and parallel, the equations are independent and the system is inconsistent. No solution exists.

Example 2: Solve the system by graphing.

\[3x + 4y = 12\]
\[-3x - 4y = -24\]

Because the equations are in general form, we graph the lines by finding the intercepts and an insurance point in a table of values.

\[
\begin{array}{|c|c|}
\hline
x & \quad y \\ \hline
0 & 3 \\ 4 & 0 \\ 8 & -3 \\
\hline
\end{array}
\quad \quad \quad \quad
\begin{array}{|c|c|}
\hline
x & \quad y \\ \hline
0 & 6 \\ 8 & 0 \\ 4 & 3 \\
\hline
\end{array}
\]

This system is inconsistent with no solution.
The equations are different so they are independent equations.

To verify that this system has no solution, show that the two lines have the same slope but different y–intercepts by converting to slope–intercept form.

\[3x + 4y = 12 \quad \rightarrow \quad 4y = -3x + 12 \quad \rightarrow \quad y = \frac{-3}{4}x + 3\]
\[-3x - 4y = -24 \quad \rightarrow \quad -4y = 3x - 24 \quad \rightarrow \quad y = \frac{-3}{4}x + 6\]
Case 3) Lines that coincide: If the lines coincide, the equations are dependent and the system is consistent. The solutions are the points on the same line so there are infinitely many solutions to the system.

Example 3: Solve the system by graphing.

\[
3x + 4y = 12 \\
-6x - 8y = -24
\]

Because the equations are in general form, we graph the lines by finding the intercepts and an insurance point in a table of values.

\[
\begin{array}{c|c}
3x + 4y = 12 & -6x - 8y = -24 \\
\hline
x & y & x & y \\
0 & 3 & 0 & 3 \\
4 & 0 & 4 & 0 \\
8 & -3 & 8 & -3 \\
\end{array}
\]

This system is consistent with an infinite number of solution, the points on the common line. The equations have the same graph so they are dependent equations. The solution set can be written as \( \{(x, y) \mid 3x + 4y = 12 \} \) or \( \{(x, y) \mid -6x - 8y = -24 \} \).

To verify that this system has an infinite number of solutions, show that the two lines have the same slope and the same y-intercepts by converting to slope-intercept form.

\[
3x + 4y = 12 \quad \rightarrow \quad 4y = -3x + 12 \quad \rightarrow \quad y = \frac{-3}{4} x + 3
\]

\[
-6x - 8y = -24 \quad \rightarrow \quad -8y = 6x - 24 \quad \rightarrow \quad y = \frac{-3}{4} x + 3
\]
The methods of addition or substitution can be used to solve systems of equations to get the same results as with graphing. Actually, a good idea is to solve the system first either by substitution or addition and then graph! Here are the same three examples done previously by graphing but this time done by either substitution or addition. I will again call them Examples 1, 2 and 3 and you can look back on the previous pages and notice that the results are the same as using graphing!

Example 1: Solve the system by substitution.

\[ x - y = 4 \quad \rightarrow \quad x = y + 4 \]
\[ 2x + y = 5 \]

I.) \[ 2(y + 4) + y = 5 \]
II.) \[ x - y = 4 \]
III.) \[ 2x + y = 5 \]

\[
\begin{align*}
2y + 8 + y &= 5 \\
x - (-1) &= 4 \\
3y + 8 &= 5 \\
x + 1 &= 4 \\
3y &= -3 \\
y &= -1
\end{align*}
\]

The solution is \((3, -1)\).

Example 2: Solve the system by addition.

\[
\begin{align*}
3x + 4y &= 12 \\
-3x - 4y &= -24
\end{align*}
\]

\[ 0 = -12 \]

When we reach a false conclusion such as \(0 = -12\) where both variables have been eliminated, it indicates that the system is inconsistent and has no solution!

To verify that this system has no solution, show that the two lines have the same slope but different \(y\)-intercepts by converting to slope–intercept form.

\[
\begin{align*}
3x + 4y &= 12 \\
-3x - 4y &= -24
\end{align*}
\]

\[
\begin{align*}
4y &= -3x + 12 \\
-4y &= 3x - 24
\end{align*}
\]

\[
\begin{align*}
y &= \frac{-3}{4} x + 3 \\
y &= \frac{-3}{4} x + 6
\end{align*}
\]

Example 3: Solve the system by addition.

\[
\begin{align*}
3x + 4y &= 12 \\
-6x - 8y &= -24
\end{align*}
\]

\[
\begin{align*}
2(3x + 4y) &= 2 \cdot 12 \\
-6x - 8y &= -24
\end{align*}
\]

\[
\begin{align*}
6x + 8y &= 24 \\
-6x - 8y &= -24
\end{align*}
\]

\[ 0 = 0 \]

When we reach a true conclusion such as \(0 = 0\) where both variables have been eliminated, it indicates that the system is consistent and has an infinite number of solutions! The equations are dependent.

To verify that this system has an infinite number of solutions, show that the two lines have the same slope and the same \(y\)-intercepts by converting to slope–intercept form.

\[
\begin{align*}
3x + 4y &= 12 \\
-6x - 8y &= -24
\end{align*}
\]

\[
\begin{align*}
4y &= -3x + 12 \\
-8y &= 6x - 24
\end{align*}
\]

\[
\begin{align*}
y &= \frac{-3}{4} x + 3 \\
y &= \frac{-3}{4} x + 3
\end{align*}
\]