1) A quadratic equation is any equation that can be written in the form \( ax^2 + bx + c = 0 \) where \( a, b, \) and \( c \) are real numbers and \( a \neq 0 \).

2) Here are a few examples of quadratic equations.
   A) \( 3x^2 + 7x + 2 = 0 \)  
   B) \( 12x^2 - 24x = 0 \)  
   C) \( 16x^2 - 32 = 0 \)  
   D) \( 25x^2 = 0 \)

The term they all have in common is the quadratic term or the term to the second power. Thus quadratic equations are also called second degree equations.

3) Many quadratic equations can be solved by factoring and then using the zero-factor property that states that if \( ab = 0 \) then either \( a = 0 \) or \( b = 0 \).

4) A way to find out if a quadratic equation can be solved by factoring is to use the test for factoribility. If the discriminant \( b^2 + (-4)ac = \) a perfect square, then the quadratic equation can be solved by factoring.

5) Can \( 3x^2 + 7x + 2 = 0 \) be solved by factoring?
   \( a = 3, \ b = 7 \) and \( c = 2 \)
   \( b^2 + (-4)ac = 7^2 + (-4)(3)(2) = 49 + (-24) = 25, \) a perfect square! Yes,
   \( 3x^2 + 7x + 2 = 0 \) can be solved by factoring.

6) Here is the quadratic equation \( 3x^2 + 7x + 2 = 0 \) solved by factoring!
   \( 3x^2 + 7x + 2 = 0 \)
   \( (x + 2) \cdot (3x + 1) = 0 \) Factor!
   either \( x + 2 = 0 \) or \( 3x + 1 = 0 \) The zero – factor property!
   \( x = -2 \) or \( x = -\frac{1}{3} \)

7) Quadratic equations can also be solved using the square root property.
   If \( c > 0 \), the equation \( x^2 = c \) has two real solutions, \( x = \sqrt{c} \) and \( x = -\sqrt{c} \) or \( x = \pm \sqrt{c} \).

8) Solve quadratic equations using the square root property if they are in the form
   \( x^2 = c \)  or \( (x \pm b)^2 = c \)  or  \( (ax \pm b)^2 = c \) where \( c > 0 \).

   a) \( x^2 = 144 \)
      \( \sqrt{x^2} = \sqrt{144} \)
      \( |x| = 12 \)
      \( x = \pm 12 \)

   b) \( (x - 4)^2 = 49 \)
      \( \sqrt{(x - 4)^2} = \sqrt{49} \)
      \( |x - 4| = 7 \)
      \( x - 4 = 7 \) or \( x - 4 = -7 \)
      \( x = 11 \) or \( x = -3 \)
c) \((2x + 5)^2 = 20\)
\[
\sqrt{(2x + 5)^2} = \sqrt{20}
\]
\[
|2x + 5| = 2\sqrt{5}
\]
\[
2x + 5 = 2\sqrt{5} \quad \text{or} \quad 2x + 5 = -2\sqrt{5}
\]
\[
2x = -5 + 2\sqrt{5} \quad \text{or} \quad 2x = -5 - 2\sqrt{5}
\]
\[
x = \frac{-5+2\sqrt{5}}{2} \quad \text{or} \quad x = \frac{-5-2\sqrt{5}}{2} \quad \text{■}
\]

Notice that a) and b) could have been solved by factoring because the solutions were rational numbers. c) could not have been solved by factoring because the solutions are irrational numbers.

9) All quadratic equations can be solved by completing the square so it is a more powerful method than factoring because factoring only works when the algebraic expression on the left or right side of the equation is factorable. To use this method, the formula for finding \(c\) and completing the square is \(c = (\frac{1}{2}b)^2\). Note that the quadratic term \(x^2\) must have a coefficient of 1!

\[
a) \quad x^2 + 12x + \boxed{36} = x^2 + 12x + \boxed{36} \quad c = (\frac{1}{2}b)^2 = \left(\frac{1}{2} \cdot 12\right)^2 = 6^2 = 36
\]
\[
b) \quad x^2 + 5x + \boxed{\frac{25}{4}} = x^2 + 5x + \boxed{\frac{25}{4}} \quad c = (\frac{1}{2}b)^2 = \left(\frac{1}{2} \cdot 5\right)^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4}
\]
\[
c) \quad x^2 - 3x + \boxed{\frac{9}{4}} = x^2 - 3x + \boxed{\frac{9}{4}} \quad c = (\frac{1}{2}b)^2 = \left(\frac{1}{2} \cdot -3\right)^2 = \left(-\frac{3}{2}\right)^2 = \frac{9}{4}
\]
\[
d) \quad x^2 - \frac{1}{2}x + \boxed{\frac{1}{16}} = x^2 - \frac{1}{2}x + \boxed{\frac{1}{16}} \quad c = (\frac{1}{2}b)^2 = \left(\frac{1}{2} \cdot -\frac{1}{2}\right)^2 = \left(-\frac{1}{4}\right)^2 = \frac{1}{16}
\]

The beauty of the method is that we are creating perfect square trinomials that when factored are equivalent to the square of a binomial.
\[
x^2 + 12x + \boxed{36} = x^2 + 12x + \boxed{36} = (x + 6)^2
\]

10) Here is the quadratic equation \(x^2 + 8x + 7 = 0\) solved by completing the square.
\[
x^2 + 8x + 7 = 0
\]
\[
x^2 + 8x + \boxed{16} = -7 + \boxed{16} \quad \text{Set up the equation so the square can be completed.}
\]
\[
x^2 + 8x + \boxed{16} = -7 + \boxed{16} \quad \text{Complete the square on the left + add the same amount on right!}
\]
\[
(x + 4)^2 = 9
\]
\[
\sqrt{(x + 4)^2} = \sqrt{9}
\]
\[
|x + 4| = 3
\]
\[
x + 4 = 3 \quad \text{or} \quad x + 4 = -3
\]
\[
x = -1 \quad \text{or} \quad x = -7 \quad \text{■}
\]

11) Here is the quadratic equation \(x^2 + 6x - 15 = 0\) solved by completing the square.
\[
x^2 + 6x - 15 = 0
\]
\[
x^2 + 6x + \boxed{9} = 15 + \boxed{9} \quad \text{Set up the equation so the square can be completed.}
\]
\[
x^2 + 6x + \boxed{9} = 15 + \boxed{9} \quad \text{Complete the square on the left + add the same amount on right!}
\]
\[
(x + 3)^2 = 24
\]
\[
\sqrt{(x + 3)^2} = \sqrt{24}
\]
\[
|x + 3| = 2\sqrt{6}
\]
\[
x + 3 = 2\sqrt{6} \quad \text{or} \quad x + 3 = -2\sqrt{6}
\]
\[
x = -3 + 2\sqrt{6} \quad \text{or} \quad x = -3 - 2\sqrt{6} \quad \text{■}
\]

Because the solutions are irrational, \(x^2 + 6x + 15 = 0\) could not be solved by factoring!
12) Here is the quadratic equation \( x^2 - 5x + 4 = 0 \) solved by completing the square.
\[
x^2 - 5x + \square = -4 + \square
\]
Set up the equation so the square can be completed.
\[
x^2 - 5x + \frac{25}{4} = -4 + \frac{25}{4}
\] Complete the square on the left + add the same amount on right!

Next multiply every term by the LCD, 4, to clear the fractions from the equation.
\[
4 \cdot x^2 - 4 \cdot 5x + 4 \cdot \frac{25}{4} = 4 \cdot (-4) + 4 \cdot \frac{25}{4}
\]
\[
4x^2 - 20x + 25 = -16 + 25
\]
\[
(2x - 5)^2 = 9 \] Factor the left side and add on the right side.
\[
\sqrt{(2x - 5)^2} = \sqrt{9}
\]
\[
|2x - 5| = 3
\]
\[
2x - 5 = 3 \text{ or } 2x - 5 = -3
\]
\[
2x = 8 \text{ or } 2x = 2
\]
\[
x = 4 \text{ or } x = 1
\]

13) Here is the quadratic equation \( 2x^2 - 5x - 3 = 0 \) solved by completing the square.
\[
\frac{2x^2}{2} - \frac{5x}{2} - \frac{3}{2} = \frac{0}{2} \]
Divid each term by the coefficient of the quadratic term.
\[
x^2 - \frac{5}{2}x - \frac{3}{2} = 0 \]
Simplify.
\[
x^2 - \frac{5}{2}x + \square = \frac{3}{2} + \square
\]
Set up the equation so the square can be completed.
\[
x^2 - \frac{5}{2}x + \frac{25}{16} = \frac{3}{2} + \frac{25}{16}
\]
Complete the square on the left + add the same amount on right!

Next multiply every term by the LCD, 16, to clear the fractions from the equation.
\[
16 \cdot x^2 - 16 \cdot \frac{5}{2}x + 16 \cdot \frac{25}{16} = 16 \cdot \frac{3}{2} + 16 \cdot \frac{25}{16}
\]
\[
16x^2 - 40x + 25 = 24 + 25
\]
\[
(4x - 5)^2 = 49 \] Factor the left side and add on the right side.
\[
\sqrt{(4x - 5)^2} = \sqrt{49}
\]
\[
|4x - 5| = 7
\]
\[
4x - 5 = 7 \text{ or } 4x - 5 = -7
\]
\[
4x = 12 \text{ or } 4x = -2
\]
\[
x = 3 \text{ or } x = \frac{-1}{2}
\]

14) All quadratic equations can also be solved by the quadratic formula. 
If \( ax^2 + bx + c = 0 \) \((a \neq 0)\), then the solutions of the equation can be found by using the formula: \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \)

15) Here is the quadratic equation \( x^2 + 8x + 7 = 0 \) solved using the quadratic formula.
\( a = 1, \ b = 8 \) and \( c = 7 \)
\[
x = \frac{-8 \pm \sqrt{8^2 + (-4)(1)(7)}}{2(1)}
\]
\[
x = \frac{-8 \pm \sqrt{64 - (-28)}}{2}
\]
\[
x = \frac{-8 \pm \sqrt{96}}{2}
\]
\[
x = \frac{-8 \pm 4}{2}
\]
High Road/Low Road
\[
x = \frac{-8 + 6}{2} \quad \text{or} \quad \frac{-8 - 6}{2}
\]
\[
x = -1 \quad \text{or} \quad -7
\]
16) Here is the quadratic equation \( x^2 + 6x - 15 = 0 \) solved using the quadratic formula. 
\[ a = 1, \ b = 6 \text{ and } c = -15 \]

\[
x = \frac{-6 \pm \sqrt{6^2 + 4(1)(-15)}}{2(1)} \\
= \frac{-6 \pm \sqrt{36 + 60}}{2} \\
= \frac{-6 \pm \sqrt{96}}{2} \\
= \frac{-6 \pm 4\sqrt{6}}{2} \\
= -3 \pm 2\sqrt{6} \\
\]

The Bermuda Triangle Technique: Divide the common factor out of 6, 4 and 2.
\[
x = \frac{-3 \pm 2\sqrt{6}}{1} \\
x = -3 \pm 2\sqrt{6} \]

17) Here is the quadratic equation \( x^2 - 5x + 4 = 0 \) solved using the quadratic formula. 
\[ a = 1, \ b = -5 \text{ and } c = 4 \]

\[
x = \frac{(-5) \pm \sqrt{(-5)^2 - 4(1)(4)}}{2(1)} \\
= \frac{5 \pm \sqrt{25 - 16}}{2} \\
= \frac{5 \pm 3}{2} \\
= \frac{5 + 3}{2} \text{ or } \frac{5 - 3}{2} \\
x = 1 \text{ or } 4 \]

18) Here is the quadratic equation \( 2x^2 - 5x - 3 = 0 \) solved using the quadratic formula. 
\[ a = 2, \ b = -5 \text{ and } c = -3 \]

\[
x = \frac{(-5) \pm \sqrt{(-5)^2 - 4(2)(-3)}}{2(2)} \\
= \frac{5 \pm \sqrt{25 + 24}}{4} \\
= \frac{5 \pm \sqrt{49}}{4} \\
= \frac{5 \pm 7}{4} \\
= \frac{5 + 7}{4} \text{ or } \frac{5 - 7}{4} \\
x = \frac{3}{2} \text{ or } -1 \]

19) Here is the quadratic equation \( 5x^2 - 7x - 2 = 0 \) solved using the quadratic formula. 
\[ a = 5, \ b = -7 \text{ and } c = -2 \]

\[
x = \frac{(-7) \pm \sqrt{(-7)^2 - 4(5)(-2)}}{2(5)} \\
= \frac{7 \pm \sqrt{49 + 40}}{10} \\
= \frac{7 \pm \sqrt{89}}{10} \\
\]

Because only 1 divides evenly into 7, 1, and 10 the Bermuda Triangle Technique cannot be used to further simplify the expression.

\[
x = \frac{7 \pm \sqrt{89}}{10} \\
\]
20) The discriminant \( b^2 + (-4)ac \) is used to predict the type of solutions without actually solving the quadratic equation.

- If \( b^2 + (-4)ac > 0 \) and a perfect square, then the solutions are rational numbers and unequal.
- If \( b^2 + (-4)ac > 0 \) but not a perfect square, then the solutions are irrational numbers and unequal.
- If \( b^2 + (-4)ac = 0 \), then the solutions are rational numbers and equal.
- If \( b^2 + (-4)ac < 0 \), then the solutions are complex conjugates.

21) To find the type of solutions of the equation \( x^2 - 2x - 3 = 0 \), determine the value of the discriminant \( b^2 + (-4)ac \).

\[
b^2 + (-4)ac = (-2)^2 + (-4)(1)(-3) = 4 + 12 = 16
\]

Because the discriminant is greater than 0 and a perfect square, the solutions will be rational numbers and unequal. The actual solutions are \( x = 3 \) or \( x = -1 \).

22) To find the type of solutions of the equation \( x^2 - 6x - 11 = 0 \), determine the value of the discriminant \( b^2 + (-4)ac \).

\[
b^2 + (-4)ac = (-6)^2 + (-4)(1)(-11) = 36 + 44 = 80
\]

Because the discriminant is greater than 0 but not a perfect square, the solutions will be irrational numbers and unequal. The actual solutions are \( x = 3 + 2\sqrt{5} \) or \( x = 3 - 2\sqrt{5} \).

23) To find the type of solutions of the equation \( x^2 - 10x + 25 = 0 \), determine the value of the discriminant \( b^2 + (-4)ac \).

\[
b^2 + (-4)ac = (-10)^2 + (-4)(1)(25) = 100 + (-100) = 0
\]

Because the discriminant equals 0, the solutions will be rational numbers and equal. The actual solutions are \( x = 5 \) or \( x = 5 \).

24) To find the type of solutions of the equation \( x^2 - 3x + 5 = 0 \), determine the value of the discriminant \( b^2 + (-4)ac \).

\[
b^2 + (-4)ac = (-3)^2 + (-4)(1)(5) = 9 + (-20) = -11
\]

Because the discriminant equals 0, the solutions will be complex conjugates. The actual solutions are \( x = \frac{3}{2} + \frac{\sqrt{11}}{2}i \) or \( x = \frac{3}{2} - \frac{\sqrt{11}}{2}i \).