Quadratic and Other Nonlinear Inequalities
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Example 1: Solve the quadratic inequality \( x^2 + 2x - 3 \geq 0 \).

**Step 1.** Factor \( x^2 + 2x - 3 \).

\[
x^2 + 2x - 3 \geq 0.
\]
\[
(x - 1)(x + 3) \geq 0
\]

**Step 2.** Find the critical points by setting the binomial factors equal to 0.

\[
x - 1 = 0 \rightarrow x = 1
\]
\[
x + 3 = 0 \rightarrow x = -3
\]

**Step 3.** Make a number line and locate the critical points on the number line. Use open circles for critical points that are not solutions and solid circles for critical points that are solutions. Determine the regions based on the critical points.

![Number line with regions labeled A, B, and C]

**Step 4.** Test a point in each region in the factored form of the inequality to find the solution set.

**In Region A. Test \(-4\)**

\[
(x - 1)(x + 3) \geq 0
\]
\[
(-4 - 1)(-4 + 3) \geq 0
\]
\[
(-5)(-1) \geq 0
\]
\[
5 \geq 0 \text{ True, therefore all the points in Region A are solutions!}
\]

**In Region B. Test 0**

\[
(x - 1)(x + 3) \geq 0
\]
\[
(0 - 1)(0 + 3) \geq 0
\]
\[
(-1)(3) \geq 0
\]
\[
-3 \geq 0 \text{ False, therefore all the points in Region B are not solutions!}
\]

**In Region C. Test 2**

\[
(x - 1)(x + 3) \geq 0
\]
\[
(2 - 1)(2 + 3) \geq 0
\]
\[
(1)(5) \geq 0
\]
\[
5 \geq 0 \text{ True, therefore all the points in Region C are solutions!}
\]
**Step 5.** Use a number line graph and interval notation to show the solution set.

The solution set in interval notation is \((-\infty, -3] \cup [1, \infty)\).

Example 2: Solve the inequality \(\frac{1}{x} \leq 6\)

**Step 1.** The right side has to equal zero. Subtract 6 from both sides and then combine the two fractions to have one fraction on the left side.

\[
\frac{1}{x} \leq 6
\]

\[
\frac{1}{x} - 6 \leq 0
\]

\[
\frac{1}{x} - \frac{6}{1} \leq 0
\]

\[
\frac{1 - 6x}{x} \leq 0
\]

**Step 2.** Find the critical points by setting the numerator and denominator equal to 0.

\[1 - 6x = 0 \rightarrow -6x = -1 \rightarrow x = \frac{1}{6}\]

\[x = 0\]

**Step 3.** Make a number line and locate the critical points on the number line. Use open circles for critical points that are not solutions and solid circles for critical points that are solutions. Determine the regions based on the critical points. Note that we use an open circle for 0 because it cannot be a solution. Zero cannot be a solution because it would cause the fraction on the left side of the inequality to be **undefined**!

![Number line with regions A, B, C]

**Step 4.** Test a point in each region in the latest fractional form of the inequality to find the solution set.

In Region A, Test \(-1\)

\[
\frac{1 - 6(-1)}{-1} \leq 0
\]

\[
\frac{-7}{-1} \leq 0
\]

\[-7 \leq 0 \text{ True, therefore all the points in Region A are solutions!}\]
In Region B, Test $\frac{1}{12}$

\[
\begin{align*}
\frac{1-6x}{x} & \leq 0 \\
\frac{1-6(\frac{1}{6})}{\frac{1}{6}} & \leq 0 \\
\frac{6}{12} & \leq 0 \\
\frac{6}{12} \div \frac{1}{12} & \leq 0 \\
\frac{6}{12} \cdot \frac{12}{1} & \leq 0 \\
6 \leq 0 & False, therefore all the points in Region B are not solutions!
\end{align*}
\]

In Region C, Test 1

\[
\begin{align*}
\frac{1-6x}{x} & \leq 0 \\
\frac{1-6(1)}{1} & \leq 0 \\
\frac{1-6}{1} & \leq 0 \\
\frac{-5}{1} & \leq 0 \\
-5 \leq 0 & True, therefore all the points in Region C are solutions!
\end{align*}
\]

**Step 5.** Use a number line graph and interval notation to show the solution set.

The solution set in interval notation is \((-\infty, 0) \cup \left[\frac{1}{6}, \infty\right)\).

Example 3: Solve the inequality \(\frac{x^2-6x+8}{x+4} \geq 0\).

**Step 1.** Factor \(x^2 - 6x + 8\).

\[
\frac{x^2-6x+8}{x+4} \geq 0
\]

\[
\frac{(x-2)(x-4)}{x+4} \geq 0
\]

**Step 2.** Find the critical points by setting the binomials in the numerator and denominator equal to 0.

\[
x - 2 = 0 \rightarrow x = 2 \\
x - 4 = 0 \rightarrow x = 4 \\
x + 4 = 0 \rightarrow x = -4
\]

**Step 3.** Make a number line and locate the critical points on the number line. Use open circles for critical points that are not solutions and solid circles for critical points that are solutions. Determine the regions based on the critical points. Note that we use an open circle for -4 because it cannot be a solution. Negative 4 cannot be a solution because it would cause the fraction on the left side of the inequality to be undefined!
Step 4. Test a point in each region in the factored form of the inequality to find the solution set.

In Region A, Test $-5$
\[
\frac{(x-2)(x-4)}{x+4} \geq 0
\]
\[
\frac{(-5-2)(-5-4)}{-5+4} \geq 0
\]
\[
\frac{(-7)(-9)}{-1} \geq 0
\]
\[
\frac{63}{1} \geq 0
\]
$-63 \geq 0$ False, therefore all the points in Region A are not solutions!

In Region B, Test 0
\[
\frac{(x-2)(x-4)}{x+4} \geq 0
\]
\[
\frac{(0-2)(0-4)}{0+4} \geq 0
\]
\[
\frac{(-2)(-4)}{4} \geq 0
\]
\[
\frac{8}{4} \geq 0
\]
$2 \geq 0$ True, therefore all the points in Region B are solutions!

In Region C, Test 3
\[
\frac{(x-2)(x-4)}{x+4} \geq 0
\]
\[
\frac{(3-2)(3-4)}{3+4} \geq 0
\]
\[
\frac{(1)(-1)}{7} \geq 0
\]
\[
\frac{-1}{7} \geq 0
\]
False, therefore all the points in Region C are not solutions!

In Region D, Test 5
\[
\frac{(x-2)(x-4)}{x+4} \geq 0
\]
\[
\frac{(5-2)(5-4)}{5+4} \geq 0
\]
\[
\frac{(3)(1)}{9} \geq 0
\]
\[
\frac{1}{3} \geq 0
\]
True, therefore all the points in Region D are solutions!

Step 5. Use a number line graph and interval notation to show the solution set.

The solution set in interval notation is $(-4, 2] \cup [4, \infty)$. 