ALGEBRAIC FRACTIONS A LA GREEN

I) Reducing Algebraic Fractions: \( \frac{ab}{cb} = \frac{a}{c} \); \( b \neq 0, c \neq 0 \)

Example 1:
\[
\frac{6x^2 + x - 2}{8x^2 + 2x - 3} = \frac{(2x - 1)(3x + 2)}{(2x - 1)(4x + 3)} = \frac{3x + 2}{4x + 3}
\]

Ex 2:
\[
\frac{2a^2 - 5ab - 3b^2}{3b^2 - ab} = \frac{(a - 3b)(2a + b)}{b(3b - a)} = \frac{-1(2a + b)}{b} = \frac{-2a - b}{b} \text{ or } \frac{-2a + b}{b}
\]

II) Addition: \( \frac{a}{b} + \frac{c}{b} = \frac{a + c}{b} \); \( b \neq 0 \)

Example 3:
\[
\frac{2x}{x + 2} + \frac{4}{x + 2} = \frac{2x + 4}{x + 2} = \frac{2(x + 2)}{(x + 2)} = 2
\]

Example 4:
\[
\frac{5}{x + 3} + \frac{7}{x - 4} = \frac{5(x - 4)}{(x + 3)(x - 4)} + \frac{7(x + 3)}{(x - 4)(x + 3)} = \frac{5x - 20 + 7x + 21}{(x + 3)(x - 4)} = \frac{12x + 1}{(x + 3)(x - 4)}
\]

III) Subtraction: \( \frac{a}{b} - \frac{c}{b} = \frac{a}{b} + \frac{-c}{b} = \frac{a + (-c)}{b} \); \( b \neq 0 \)

Example 5:
\[
\frac{5}{c - 10} - \frac{10}{c - 10} = \frac{5}{c - 10} + \frac{-10}{c - 10} = \frac{5 + (-10)}{c - 10} = \frac{-5}{c - 10}
\]

Example 6:
\[
\frac{8}{c - 4} - \frac{4}{4 - c} = \frac{8}{c - 4} + \frac{-4}{4 - c} = \frac{8}{c - 4} + \frac{(-1)(-4)}{c - 4} = \frac{8}{c - 4} + \frac{4}{c - 4} = \frac{12}{c - 4}
\]

Example 7:
\[
\frac{12}{a - 2} - \frac{a + 3}{a - 5} = \frac{12}{a - 2} + \frac{-1(a + 3)}{a - 5} = \frac{12(a - 5)}{(a - 2)(a - 5)} + \frac{-1(a + 3)(a - 2)}{(a - 2)(a - 5)} = \to
\]
\[
\frac{12a - 60 + -1(a^2 + a - 6)}{(a - 2)(a - 5)} = \frac{12a - 60 - a^2 - a + 6}{(a - 2)(a - 5)} = \frac{-a^2 + 11a - 54}{(a - 2)(a - 5)}
\]
IV) Multiplication: \( \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} ; b \neq 0, d \neq 0 \)

Example 8: \( \frac{2p^2 - 5p - 3}{p^2 - 9} \cdot \frac{2p^2 + 5p - 3}{2p^2 + 5p + 2} = \frac{(p - 3)(2p + 1)(p + 3)(2p - 1)}{(p - 3)(p + 3)(p + 2)(2p + 1)} = \frac{2p - 1}{p + 2} \)

V) Division: \( \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc} ; b \neq 0, c \neq 0, d \neq 0 \)

Example 9:
\[
\frac{2x^2 + 3xy + y^2}{y^2 - x^2} \cdot \frac{6x^2 + 5xy + y^2}{2x^2 - xy - y^2} = \frac{2x^2 + 3xy + y^2}{y^2 - x^2} \cdot \frac{2x^2 - xy - y^2}{6x^2 + 5xy + y^2} = \rightarrow \\
\frac{(x + y)(2x + y)(x - y)(2x + y)}{(y - x)(y + x)(2x + y)(3x + y)} = -\frac{1(2x + y)}{3x + y} = \frac{-2x - y}{3x + y} \\
\]

VI) Complex Fractions:

Example 10:
\[
\frac{1 + \frac{6}{x} + \frac{8}{x^2}}{1 + \frac{1}{x} - \frac{12}{x^2}} = \frac{x^2 \left( 1 + \frac{6}{x} + \frac{8}{x^2} \right)}{x^2 \left( 1 + \frac{1}{x} - \frac{12}{x^2} \right)} = \frac{x^2 \cdot \frac{1}{1} + x^2 \cdot \frac{6}{x} + x^2 \cdot \frac{8}{x^2}}{x^2 \cdot \frac{1}{1} + x^2 \cdot \frac{1}{x} - x^2 \cdot \frac{12}{x^2}} = \rightarrow \\
\]

\[
\frac{x^2 + 6x + 8}{x^2 + x - 12} = \frac{(x + 2)(x + 4)}{(x - 3)(x + 4)} = \frac{(x + 2)}{(x - 3)} \\
\]

Example 11:
\[
\frac{x + y}{x^{-1} + y^{-1}} = \frac{\frac{x}{1} + \frac{1}{y}}{\frac{1}{x} + \frac{1}{y}} = \frac{xy \left( \frac{x}{1} + \frac{1}{y} \right)}{xy \left( \frac{1}{x} + \frac{1}{y} \right)} = \frac{xy \cdot \frac{1}{1} + xy \cdot \frac{1}{1}}{xy \cdot \frac{1}{x} + xy \cdot \frac{1}{y}} = \rightarrow \\
\]

\[
\frac{x^2y + xy^2}{y + x} = \frac{xy(x + y)}{(y + x)} = xy \\
\]
VII) Equations Containing Algebraic Fractions: Multiply each fraction by the Least Common Denominator of ALL of the fractions in the equation. Remember to note values that may have to be discarded as they are extraneous solutions and thus will not be in your set of solutions.

Example 12: \[ \frac{6}{x-6} + 3 = \frac{x}{x-6}; \] Note that 6 cannot be a solution. \( x \neq 6 \)

\[(x - 6) \cdot \frac{6}{x-6} + (x - 6) \cdot 3 = (x - 6) \cdot \frac{x}{x-6} \rightarrow 6 + 3x - 18 = x \rightarrow 3x - 12 = x \rightarrow 3x - x = 12 \rightarrow 2x = 12 \rightarrow x = 6 \]

However 6 is an extraneous and must be discarded so this equation has NO SOLUTION!

Example 13: \[ \frac{u - 2}{u - 3} + 3 = u + \frac{4 - u}{u - 3}; \] Note that 3 cannot be a solution. \( x \neq 3 \)

\[(u - 3) \cdot \frac{u - 2}{u-3} + (u - 3) \cdot 3 = (u - 3) \cdot u + (u - 3) \cdot \frac{4 - u}{u-3} \]

\[u - 2 + 3u - 9 = u^2 - 3u + 4 - u \rightarrow 4u - 11 = u^2 - 4u + 4 \]
\[0 = u^2 - 4u - 4u + 4 + 11 \rightarrow 0 = u^2 - 8u + 15 \]
\[0 = (u - 3)(u - 5) \rightarrow \text{either } u - 3 = 0 \text{ or } u - 5 = 0 \]
\[u = 3 \text{ or } u = 5, \text{ however 3 is an extraneous solution and must be discarded so the only solution is 5. } u = 5 \text{ only!} \]

Example 14: Hillary can clean her house in New York in 3 hours while if she is off campaigning for senator and Bill has to clean the house it would take him 10 hours... How long would it take them to clean their house if they worked together?

Let \( x = \# \) of hours for them to clean the house working together

\[ \text{part of job Hillary} + \text{part of job Bill} = \text{part of job they do together} \]
\[ \text{does in one hour} + \text{does in one hour} = \text{in one hour} \]
\[ \frac{1}{3} + \frac{1}{10} = \frac{1}{x} \]

\[30x \cdot \frac{1}{3} + 30x \cdot \frac{1}{10} = 30x \cdot \frac{1}{x} \rightarrow 10x + 3x = 30 \rightarrow 13x = 30 \]
\[x = \frac{30}{13} = 2\frac{4}{13} \text{ hours} \]

It would take them \( 2\frac{4}{13} \text{ hours} \) for them to clean the house working together!
Example 15: A woman who can row 3 mph in still water rows 10 miles downstream on the Green River and returns upstream in a total of 12 hours. Find the speed of the current.

Let \( x = \) the speed of the current

<table>
<thead>
<tr>
<th>trip</th>
<th>distance</th>
<th>rate</th>
<th>time</th>
</tr>
</thead>
<tbody>
<tr>
<td>rowing downstream</td>
<td>10</td>
<td>( 3 + x )</td>
<td>( \frac{10}{3 + x} )</td>
</tr>
<tr>
<td>rowing upstream</td>
<td>10</td>
<td>( 3 - x )</td>
<td>( \frac{10}{3 - x} )</td>
</tr>
</tbody>
</table>

\[
(3 + x)(3 - x) \cdot \frac{10}{3 + x} + (3 + x)(3 - x) \cdot \frac{10}{3 - x} = (3 + x)(3 - x) \cdot 12 \quad \rightarrow \\
10(3 - x) + 10(3 + x) = 12 \cdot (9 - x^2) \quad \rightarrow \\
30 - 10x + 30 + 10x = 108 - 12x^2 \\
60 = 108 - 12x^2 \quad \rightarrow \\
12x^2 + 60 - 108 = 0 \quad \rightarrow \\
12x^2 - 48 = 0 \\
x^2 - 4 = 0 \quad \rightarrow \\
(x - 2)(x + 2) = 0 \quad \rightarrow \\
x = 2 \text{ or } x = -2 \text{ (Reject } -2 \text{ Not possible!)}

The speed of the current is 2 miles per hour. \( \blacksquare \)