Supplement to Section 6.4 and 6.5

In this section starting on page 390 the book introduces the notion of left sum and right sum. The left sum of \( n \) rectangles is denoted by \( L_n \). The right sum of \( n \) rectangles is denoted by \( R_n \). It is left to the student to interpret from the pictures the meanings and definitions of these sums.

Informal definition: Consider the function \( f \) represented by the following graph:

![Graph](image)

This point has coordinates \((x, y)\) where \( y = f(x)\).

Notice that the four rectangles extend from the \( x\)-axis to the curve. The rectangles touch the curve on their right side and the height of the rectangle is equal to the height of the curve at this point. Thus the height of the rectangles in the pictures is \( f(x) \) (coordinate on right side). The sum of the area of these four rectangles would be called \( R_4 \). If the rectangles touched the curve on their left side, the sum would be \( L_4 \).

The formal definition is not given until Section 6.5.

**Definition Left sum and Right sum:**

If we partition the interval \([a, b]\) into \( n \) equal subintervals of length \( \Delta x = (b - a) / n \) with endpoints \( a = x_0, x_1, \ldots, x_n = b \) where \( x_k = a + k \cdot \Delta x \), then define

\[
\begin{align*}
L_n &= \text{left sum} = \sum_{k=1}^{n} f(x_{k-1}) \Delta x = \left[ f(x_0) + f(x_1) + \ldots + f(x_{n-1}) \right] \Delta x \\
R_n &= \text{right sum} = \sum_{k=1}^{n} f(x_k) \Delta x = \left[ f(x_1) + f(x_2) + \ldots + f(x_n) \right] \Delta x
\end{align*}
\]

Remark: It is sometimes easier to use

\[
L_n = \text{left sum} = \sum_{k=0}^{n-1} f(x_k) \Delta x = \left[ f(x_0) + f(x_1) + \ldots + f(x_{n-1}) \right] \Delta x
\]

**Example 1:** Compute the left sum of \( f(x) = e^{x^2} \) using 5 equal subintervals of \([0,1]\)

Solution 1) We are given that \( a = 0 \), \( b = 1 \), and \( n = 5 \).

Step 1) Compute width of each rectangle:

Since we want equal subintervals each rectangle has a width of
\[
\Delta x = (b - a) / n = (1 - 0) / 5 = 1/5.
\]
Step 2) Compute the endpoints of each interval.
We then have a partition of the interval \([0,1]\) of
\[
\begin{align*}
x_0 &= 0, \quad x_1 = 0 + 1 \left( \frac{1}{5} \right), \quad x_2 = 2 \left( \frac{1}{5} \right), \quad x_3 = 3 \left( \frac{1}{5} \right), \quad x_4 = 4 \left( \frac{1}{5} \right), \quad x_5 = 5 \left( \frac{1}{5} \right) = 1.
\end{align*}
\]

Step 3) Evaluate sum
Since we want the left sum we are going use the left five numbers. (We will not use \(x_5\))
\[
\begin{align*}
L_5 &= \sum_{k=1}^{5} f(x_{k-1}) \Delta x = \left[ f(x_0) + f(x_1) + \ldots + f(x_4) \right] \Delta x \\
&= \left[ f(0) + f(1/5) + f(2/5) + f(3/5) + f(4/5) \right] \left( \frac{1}{5} \right) \\
&= \left[ e^0 + e^{1/25} + e^{4/25} + e^{9/25} + e^{16/25} \right] \left( \frac{1}{5} \right) \\
&\approx 1.309
\end{align*}
\]

Example 2: Compute the right sum of \(f(x) = \sqrt{9-x^2}\) using 8 equal subintervals of \([-1,3]\)

Solution 2) We are given that \(a = -1\), \(b = 3\), and \(n = 8\).

Step 1) Compute width of each rectangle:
Since we want equal subintervals each rectangle has a width of
\[
\Delta x = (b - a) / n = (3 - (-1)) / 8 = 1/2.
\]

Step 2) Compute the endpoints of each interval.
We then have a partition of the interval \([-1,3]\) with \(x_k = a + k \Delta = -1 + k \left( \frac{1}{2} \right)\)
\[
\begin{align*}
x_0 &= -1, \quad x_1 = -\frac{1}{2}, \quad x_2 = 0, \quad x_3 = \frac{1}{2}, \quad x_4 = 1, \quad x_5 = \frac{3}{2}, \quad x_6 = 2, \quad x_7 = \frac{5}{2}, \quad x_8 = 3.
\end{align*}
\]

Step 3) Evaluate sum
Since we want the right sum we are going use the right eight numbers. (We will not use \(x_0\))
\[
\begin{align*}
R_8 &= \sum_{k=1}^{8} f(x_k) \Delta x = \left[ f(x_1) + \ldots + f(x_8) \right] \Delta x \\
&= \left[ f\left( -\frac{1}{2} \right) + f\left( \frac{1}{2} \right) + f(1) + f\left( \frac{3}{2} \right) + f\left( \frac{5}{2} \right) + f(3) \right] \left( \frac{1}{2} \right) \\
&= \left[ \frac{\sqrt{35}}{4} + \sqrt{9} + \frac{\sqrt{35}}{4} + \sqrt{8} + \frac{\sqrt{27}}{4} + \sqrt{5} + \frac{\sqrt{11}}{4} + \sqrt{0} \right] \left( \frac{1}{2} \right) \\
&\approx 2.280
\end{align*}
\]
The left and right sums computed above are a special case of a Riemann Sum. Riemann Sums do not require that the rectangle use the height on the left side or the height on the right side, but instead allow the height to be taken at any point in between.

**Definition: Riemann Sum**

Let \( f \) be a continuous function defined on the closed interval \([a, b]\), and let

1. \( a = x_0 < x_1 < \ldots < x_n = b \)
2. \( \Delta x_k = x_k - x_{k-1} \) for \( k = 1, 2, \ldots, n \)
3. Let \( c_k \) satisfy \( x_{k-1} \leq c_k \leq x_k \) for \( k = 1, 2, \ldots, n \)

Then the sum

\[
\sum_{k=1}^{n} f(c_k) \Delta x = f(c_1) \Delta x_1 + f(c_2) \Delta x_2 + \cdots + f(c_n) \Delta x_n
\]

is called a Riemann Sum.

In the first three sections of chapter 6, the concept of indefinite integrals and anti-derivatives was introduced. In section 4 (page 389) the definite integral symbol was used, but no formal definition was given. The formal definition of the definite integral involves limits of a sum. Justification for why a similar symbol is used for these seemingly unrelated topics is discussed in section 5 (top of page 411).

**Definition: (page 406) Definite Integral of \( f \) from \( a \) to \( b \)**

Let \( f \) be a continuous function defined on the closed interval \([a, b]\), and let

4. \( a = x_0 < x_1 < \ldots < x_n = b \)
5. \( \Delta x_k = x_k - x_{k-1} \) for \( k = 1, 2, \ldots, n \)
6. \( \Delta x_k \to 0 \) as \( n \to \infty \)
7. Let \( c_k \) satisfy \( x_{k-1} \leq c_k \leq x_k \) for \( k = 1, 2, \ldots, n \)

Then

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(c_k) \Delta x
\]

\[
= \lim_{n \to \infty} \left[ f(c_1) \Delta x_1 + f(c_2) \Delta x_2 + \cdots + f(c_n) \Delta x_n \right]
\]

is called a **definite integral** of \( f \) from \( a \) to \( b \). The **integrand** is \( f(x) \), the **lower limit** is \( a \), and the **upper limit** is \( b \).

If a Riemann Sum is computed and no limit is taken then we say that the Riemann Sum is an estimate of the definite integral.

**Example 3**: Estimate \( \int_{0}^{1} e^x \, dx \) using five left subintervals.

**Solution 3)** This is example 1. So \( \int_{0}^{1} e^x \, dx \approx 1.309 \)
Example 4: Estimate \( \int_{1}^{3} \frac{5x + 2}{x^3 + 4} \, dx \) using five right subintervals.

Solution 4) We are given that \( a = 1, b = 3, \) and \( n = 5 \) and \( f(x) = \frac{5x + 2}{x^3 + 4} \).

Step 1) Compute width of each rectangle:
Since we want equal subintervals each rectangle has a width of \( \Delta x = \frac{b - a}{n} = \frac{2}{5} = \frac{1}{5} \).

Step 2) Compute the endpoints of each interval.
We then have a partition of the interval \([1, 3]\) with
\[ x_k = a + k\Delta = 1 + k\left(\frac{1}{5}\right) \quad x_0 = 1, \quad x_1 = \frac{7}{5}, \quad x_2 = \frac{9}{5}, \quad x_3 = \frac{11}{5}, \quad x_4 = \frac{13}{5}, \quad x_5 = \frac{15}{5} = 3. \]

Step 3) Evaluate sum
Since we want the right sum we are going use the right five numbers. (We will not use \( x_0 \))
\[ R_5 = \sum_{k=1}^{5} f(x_k)\Delta x = [f(x_1) + \ldots + f(x_5)]\Delta x \]
\[ = \left[ f\left(\frac{7}{5}\right) + f\left(\frac{9}{5}\right) + f\left(\frac{11}{5}\right) + f\left(\frac{13}{5}\right) + f(3) \right]\left(\frac{1}{5}\right) \]
\[ = \left[ \frac{375}{281} + \frac{1375}{1229} + \frac{1625}{1831} + \frac{625}{899} + \frac{17}{31} \right]\left(\frac{1}{5}\right) \]
\[ \approx 0.917 \]

Step 4) Answer Question:
\[ \int_{1}^{3} \frac{5x + 2}{x^3 + 4} \, dx \approx 0.917 \]

In general the estimates are pretty bad unless a large number of subintervals are used.
Example 5: Consider \( \int \limits_0^4 3x^2 \, dx \)

a) Estimate this indefinite integral using \( n \) equal subintervals with left endpoints \( L_n \).

b) Compute the indefinite integral using the definition. In other words compute the limit of the Riemann Sum as the number of subintervals increases without bound.

Solution 5a) We are given that \( a = 0 \), \( b = 4 \), and \( f(x) = 3x^2 \).

Step 1) Compute width of each rectangle:

Since we want equal subintervals each rectangle has a width of

\[ \Delta x = \frac{b - a}{n} = \frac{4 - 0}{n} = \frac{4}{n} \]

Step 2) Compute the endpoints of each interval.

We then have a partition of the interval \([0,4]\) with \( x_k = a + k \Delta = 0 + k \left( \frac{4}{n} \right) \)

Step 3) Evaluate sum

Since we want the left sum we are going use the left \( n \) numbers. (We will not use \( x_n \))

\[
L_n = \sum_{k=1}^{n} f(x_{k-1}) \Delta x
\]

\[
= \sum_{k=1}^{n} f \left( (k-1) \frac{4}{n} \right) \left( \frac{4}{n} \right) = \sum_{k=1}^{n} 3 \left( (k-1) \frac{4}{n} \right)^2 \left( \frac{4}{n} \right)
\]

\[
= \sum_{k=1}^{n} \frac{192}{n^2} (k-1)^2
\]

\[
= \frac{192}{n^3} \sum_{k=1}^{n} (k^2 - 2k + 1)
\]

\[
= \frac{192}{n^3} \left[ \sum_{k=1}^{n} k^2 - 2 \sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \right]
\]

\[
= \frac{192}{n^3} \left[ \frac{n(n+1)(2n+1)}{6} - \frac{2n(n+1)}{2} + n \right]
\]

\[
= \frac{32n(n+1)(2n+1)}{n^3} - \frac{192n(n+1)}{n^3} + \frac{192}{n^2}
\]

This last equality came from the handout on formulas needed for section 6.5 and 6.6.
Solution 5b) We are asked to compute \( \int_{0}^{4} 3x^2 \, dx = \lim_{n \to \infty} L_n \). Using 5a) we have

\[
\int_{0}^{4} 3x^2 \, dx = \lim_{n \to \infty} L_n = \lim_{n \to \infty} \frac{32n(n+1)(2n+1)}{n^3} - \frac{192n(n+1) + 192}{n^2}
\]

\[
= 64
\]

Example 6: Compute \( \int_{0}^{2} (4x^3 - 3x + 2) \, dx \) using the limit of a left Riemann Sum.

Solution 6)

Step 1) Compute width of each rectangle:
Since we want equal subintervals each rectangle has a width of
\( \Delta x = (b - a) / n = (2 - 0) / n = 2 / n \).

Step 2) Compute the endpoints of each interval.
We then have a partition of the interval \([0,2]\) with \( x_k = a + k \cdot \Delta x = 0 + k \cdot \left( \frac{2}{n} \right) \)

Step 3) Evaluate sum
Since we want the left sum we are going use the left \( n \) numbers. (We will not use \( x_n \))

\[
L_n = \sum_{k=1}^{n} f(x_{k-1}) \Delta x
\]

\[
= \sum_{k=1}^{n} \left[ 4 \left( x_k \right)^3 - 3 \left( x_k \right) + 2 \right] \cdot \left( \frac{2}{n} \right)
\]

\[
= \sum_{k=1}^{n} \left[ 4 \left( \frac{2k}{n} \right)^3 - 3 \left( \frac{2k}{n} \right) + 2 \right] \cdot \left( \frac{2}{n} \right)
\]

\[
= \frac{64}{n^4} \sum_{k=1}^{n} k^3 - \frac{12}{n^2} \sum_{k=1}^{n} k + \frac{2}{n} \sum_{k=1}^{n} 2
\]

\[
= \frac{64}{n^4} \left( \frac{n(n+1)}{2} \right)^2 - \frac{12}{n^2} \frac{n(n+1)}{2} + \frac{2}{n} (2n)
\]
Step 4) Compute limit

\[
\int_0^2 (4x^3 - 3x + 2) \, dx = \lim_{n \to \infty} L_n
\]

\[
= \lim_{n \to \infty} \frac{64}{n^4} \left( \frac{n(n+1)}{2} \right)^2 - \frac{12}{n^2} \frac{n(n+1)}{2} + \frac{2}{n} (2n)
\]

\[
= 16 - 6 + 4 = 14
\]

Supplemental Homework:

1. Estimate \( \int_0^1 x^3 \, dx \) using three left equal subintervals.

2. Estimate \( \int_{-2}^1 \sqrt{20 - 5x} \, dx \) using six right equal subintervals.

3. Estimate \( \int_0^4 (5x^3 - 6x^2 + 7) \, dx \)
   a. Using four right equal subintervals.
   b. Using eight left equal subintervals.
   c. Using \( n \) right equal subintervals.
   d. Compute the indefinite integral as the limit of the Reimann Sum of part c above. [i.e., let the number of subintervals increase without bound \( (n \to \infty) \)]

4. Estimate \( \int_0^8 (x^2 + 4) \, dx \)
   a. Using four right equal subintervals.
   b. Using eight left equal subintervals.
   c. Using \( n \) left equal subintervals.
   d. Compute the indefinite integral as the limit of the Reimann Sum of part c above. [i.e., let the number of subintervals increase without bound \( (n \to \infty) \)]

Answers:

1. a. 1/9  
   b. 13.758
   c. 3.348  
   d. 336

3. a. 348  
   b. 336
   c. \( 320 \left( \frac{n+1}{n} \right)^2 - 64 \left[ \frac{(n+1)(2n+1)}{n^2} \right] + 28 \)
   d. 220

4. a. 272  
   b. 172
   c. \( \frac{256(n+1)(2n+1)}{3n^2} - \frac{512(n+1)}{n^2} + \frac{512}{n^2} + 32 \)
   d. \( 202 \frac{7}{3} \)
Formulas

1) \[ \sum_{k=1}^{n} c = cn \]

2) \[ \sum_{k=1}^{n} (a_n \pm b_n) = \sum_{k=1}^{n} a_n \pm \sum_{k=1}^{n} b_n \]

3) \[ \sum_{k=1}^{n} c a_n = c \cdot \sum_{k=1}^{n} a_n \]

4) \[ 1 + 2 + 3 + \cdots + n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \]

5) \[ 1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \]

6) \[ 1^3 + 2^3 + 3^3 + \cdots + n^3 = \sum_{k=1}^{n} k^3 = \left( \frac{n(n+1)}{2} \right)^2 \]