Example: Let $f(x) = \sqrt{x}$. Show $\lim_{x \to 4} \sqrt{x} = 2$.

Say $\varepsilon = 1$.

We must find $\delta$ so that $|f(x) - 2| < 1 = \varepsilon$ whenever $|x - 4| < \delta$.

$\delta = 4 - 1 = 3$ works.

We can see $\delta = 3$ works by showing a graph. If, for example, $x = 1.1$, then the graph below shows $|f(1.1) - 2| < 1 = \varepsilon$.

We could verify $\delta = 3$ works by using algebra.

The same example, $x = 1.1$, gives

$$|f(1.1) - 2| = |1.049 - 2| = |-0.951| = 0.951 < 1 = \varepsilon.$$
Actually, $\delta = 3$ is the largest possible $\delta$. Smaller $\delta$'s would also work.

Now we'll make $\varepsilon$ smaller.

Say $\varepsilon = \frac{1}{2}$.

$\delta = 4 - 2.25 = 1.75$ works.
Handout on limits.

Say $\varepsilon = 1/10$.

Then $\delta = 4 - 3.61 = 0.39$ works.

Can we generalize so that we don't have to continue making $\varepsilon$ smaller, picking exact numbers for $\varepsilon$ each time?

Can we write $\varepsilon$ as the letter "$\varepsilon$" and assume it being fixed for smaller and smaller real numbers? Yes.

The picture for general $\varepsilon$:
Let $\varepsilon$ be an arbitrary positive real number. Although $\varepsilon$ is arbitrary, we imagine it fixed while we search for a $\delta$ that would satisfy $|f(x) - 2| < \varepsilon$ whenever $|x - 4| < \delta$.

**Find a $\delta$ using a graph.**

**Observations:**

| $2 + \varepsilon$ | $f(x)$ | $|f(x) - 2| < \varepsilon$ |
|-------------------|--------|--------------------------|
| $2$               |        |                          |
| $2 - \varepsilon$ |        |                          |

| $2 + \varepsilon$ | $f(x)$ | $|f(x) - 2| \neq \varepsilon$ |
|-------------------|--------|-----------------------------|
| $2$               |        |                             |
| $2 - \varepsilon$ |        |                             |

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**Scratch:**

\[ y = \sqrt{x}, \quad y = \sqrt{x} \]
\[ 2 - \varepsilon = \sqrt{x}, \quad (2 + \varepsilon) = \sqrt{x} \]
\[ (2 - \varepsilon)^2 = x, \quad (2 + \varepsilon)^2 = x \]

The largest possible $\delta$ will be the smaller of the two distances shown along the $x$-axis (domain). It is

\[ |4 - (2 - \varepsilon)^2| = |4 - (4 - 4\varepsilon + \varepsilon^2)| = |4\varepsilon - \varepsilon^2|. \]

Let $\delta = |4\varepsilon - \varepsilon^2|$. 
The reader should verify that $|f(x) - 2| < \varepsilon$ whenever $|x - 4| < \delta$. A larger $\delta$ would be too big for then the open interval about 4 would open too far to the left.

The problem is finished, but now we'll find a $\delta$ again using another technique.

Find a $\delta$ using algebra

$|f(x) - 2| = |\sqrt{x} - 2|$

$= \left| \frac{(\sqrt{x} - 2)(\sqrt{x} + 2)}{\sqrt{x} + 2} \right|$

(trick to make $x - 4$ appear)
Handout on Limits

\[ \frac{|x-4|}{\sqrt{x^2+2}} \]

\[ \leq \frac{|x-4|}{2} \]

\[ \leq \frac{\delta}{2} \]

\[ \leq \frac{2 \cdot \varepsilon}{2} \]

\[ \leq \varepsilon \]

There is a strict inequality in the steps so we may conclude

\[ |f(x) - 2| < \varepsilon \quad \text{whenever} \quad |x-4| < \delta \quad (\delta = 2 \varepsilon). \]