Graphical $\varepsilon$-$\delta$ Proofs of Limits

If the limit does exist.

1. **Graph**: sketch $f(x)$.
2. **Graph**: mark $L$ (on the y-axis).
3. **State**: "Let $\varepsilon > 0$, $\delta$ arbitrary."
   
   **Graph**: draw the lines $y = L + \varepsilon$ and $y = L - \varepsilon$.
4. **State**: "We can choose $\delta > 0$ so that ordered pairs $(x, f(x))$ lie inside the $\varepsilon$-$\delta$ rectangle for any $x$ in $(a-\delta, a) \cup (a, a+\delta)$.
   
   **Graph**: choose $\delta$ small enough so that ordered pairs lie inside the $\varepsilon$-$\delta$ rectangle for any $x$ in $(a-\delta, a+\delta)$, $x \neq a$. Sketch $x = a-\delta$, $x = a+\delta$.
   
   In other words, choose $\delta$ small enough so that $f(x)$'s lie inside the interval $(f(a) - \varepsilon, f(a) + \varepsilon)$ on y-axis for any $x$ in $(a-\delta, a+\delta)$.
5. **State**: "$\varepsilon$ was arbitrary, so even if $\varepsilon$ were made smaller, we could find $\delta$ so that ordered pairs all lie inside the $\varepsilon$-$\delta$ rectangle."

If the limit does not exist.

1. **Graph**: sketch $f(x)$.
2. **Graph**: mark a best-guessed $L$ (on the y-axis).
3. **State**: "Let $\varepsilon > 0$, $\delta$ arbitrary."
   
   **Graph**: Draw lines $y = L - \varepsilon$ and $y = L + \varepsilon$. Draw them close enough together so that some ordered pairs will not lie inside the $\varepsilon$-$\delta$ rectangle.
4. **State**: "Choose $\delta > 0$ in an attempt to make ordered pairs lie inside $\varepsilon$-$\delta$-rect."
   
   **Graph**: Draw the lines $x = a-\delta$ and $a+\delta$.
5. **State**: "The ordered pairs do not lie inside the $\varepsilon$-$\delta$ rectangle for all $x$'s in the interval $(a-\delta, a+\delta)$, $x \neq a$.
   
   **Graph**: show the "bad" ordered pairs.
6. **State**: "No matter how small $\delta$ is made, not all ordered pairs will lie inside the $\varepsilon$-$\delta$ rectangle."
7. **State**: "We showed that for a particular $\varepsilon > 0$, there was no $\delta > 0$ that would make all ordered pairs lie inside the $\varepsilon$-$\delta$ rectangle."
8. **State**: "Any other choice of $L$ would result in the same trouble."