Show all work. Do not use a graphing calculator on problems 1, 2.

1. Find local and absolute extrema of \( f(x) = x \sqrt{x + 6} \) on the interval \([-6, 1]\).

2. Find the intervals of concavity and inflection points for
\[ f(x) = 2x^3 - 9x^2 - 24x + 16. \]

3. Let \( g(x) = \frac{x}{x+1} \).

(a) Verify that \( g(x) \) satisfies the hypothesis of the Mean Value Theorem on the interval \([0, 3]\).

(b) Find all numbers \( c \) satisfying the conclusion of MVT on \([0, 3]\).

(c) Apply the conclusion of MVT to \( g(x) \) on the interval \([-3, 0]\) to show no \( c \) exists satisfying the conclusion.

(d) Explain why part (c) does not contradict the MVT.

(e) Verify, part (c) using the graph of \( g(x) \). That is, demonstrate on the graph that no \( c \) exists.

4. Evaluate the limit, \( \lim_{x \to 1} \frac{x^a - 1}{\sin(\pi x)} \) \((a \text{ is a real number, } a \neq 0)\)

5. Let \( h(x) = \frac{x^3 - 1}{\sin(\pi x)} \).

(a) Use a graphing calculator to graph \( h(x) \) on the interval \([0, 4]\).

(b) List the vertical asymptotes.

(c) Use the graph to estimate intervals of increase/decrease.
6. Evaluate the limit using L'Hospital's Rule. \[ \lim_{x \to \infty} (\ln x)^{\frac{1}{x}}. \]

7. Let \( f'(x) = \frac{1}{3\sqrt{x}} - \sin x \). Find \( f(x) \).

8. A box with a square base and closed top has a volume of 27 cm\(^3\). Material for the base and top costs $5 per square cm. For the sides the cost is $10 per square cm. Find the cost of materials for the least expensive box.

**Extra Credit:** On an interval \( I \), assume \( f \) increases, \( g \) increases, \( f < 0 \), and \( g < 0 \). Prove \( f \cdot g \) decreases on \( I \).
1. \( f(x) = x \sqrt{x+6} \) in \([-6, 1]\)
   \[ f'(x) = \frac{2x + 6}{2 \sqrt{x+6}} + \frac{x}{2 \sqrt{x+6}} \]
   \[ = \frac{2(x+6) + x}{2 \sqrt{x+6}} \]
   \[ = \frac{3x + 12}{2 \sqrt{x+6}} \]
   \[ f'(x) = 0 \quad f'(x) \text{ is DNE} \]
   \[ 3x + 12 = 0 \quad x+6 = 6 \]
   \[ x = -4 \quad x = -6 \]
   Critical pts are \(-4, -6\).

2. \( f(x) = 2x^3 - 9x^2 - 24x + 16 \)
   \[ f'(x) = 6x^2 - 18x - 24 \]
   \[ f''(x) = 12x - 18 \]
   \[ f''(x) = 0 \quad 6(2x-3) = 0 \]
   \[ x = \frac{3}{2} \]
   \( x = \frac{3}{2} \) is a candidate for an inflection pt.

Interval:
<table>
<thead>
<tr>
<th>Interval</th>
<th>(-(\infty), (\frac{3}{2}))</th>
<th>((\frac{3}{2}, \infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test Sign</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>Sign (f'')</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>(f)</td>
<td>down, up</td>
<td></td>
</tr>
</tbody>
</table>

2. \[ \left(\frac{3}{2}\right) \]
   \[ (\frac{3}{2}, 1) \]
   Can evaluate \(f(\frac{3}{2})\) using synthetic division.

\[
\begin{array}{c|cccc}
\frac{3}{2} & 2 & -9 & -24 & 16 \\
\hline
& 2 & -6 & -33 & -67/2
\end{array}
\]

\[ f\left(\frac{3}{2}\right) = -\frac{67}{2} + 1 \]

3. \[ g(x) = \frac{x}{x+1} \]
   \[ g'(x) = \frac{(x+1) \cdot 1 - x \cdot (1+0)}{(x+1)^2} = \frac{1}{(x+1)^2} \]
   \[ g(x) \text{ is continuous on } (-\infty, 1) \cup (-1, \infty) \]
   and so is continuous on \([0, 3]\), a subset.

(a) \(g(x)\) is differentiable on \((0, 1) \cup (-1, \infty)\), and so is diff on the subset \([0, 3]\).

(b) \[ \frac{f(b) - f(a)}{b - a} = f'(c) \]
3 cont.
\[
\frac{5}{3} - 0 = \frac{1}{(c+1)^2} \\
\rightarrow (c+1)^2 = 4 \\
\rightarrow c+1 = \pm 2 \\
\rightarrow c = 1, -3 \\
\text{Only } c = -3 \text{ is in } [0, 3].
\]
\[g(c) = \frac{g(0) - g(-3)}{0 - (-3)} = g'(c) \]
4 \[g(x) \text{ does not satisfy the hypothesis of MVT on } [3, 0]. \]
\[g(x) \text{ is not cont. on } [3, 0]. \]
\[\text{No real } c \text{ exists.} \]
\[g(0) \]
\[\lim_{x \to 1} \frac{x^3 - 1}{\sin(\pi x)} \]
\[= \lim_{x \to 1} \frac{x - 1}{\sin(\pi x)} \cdot x \]
\[= \frac{\pi \cdot (-1)}{\pi} = -\frac{1}{\pi} \]
11 \[(5) \quad h(x) = \frac{x^3 - 1}{\sin(\pi x)} \]
\[\lim_{x \to 1} h(x) = -\frac{3}{\pi} \]
(c) Trace options on calculator estimates 1.5
\[h \text{ increases on } (0, 0.7) \cup (2.27, 3) \cup (3, 4) \]
\[h \text{ decreases on } (0.7, 2) \cup (2.27, 3) \cup (3, 4). \]
6 \[\lim_{x \to \infty} (\ln(x))^{1/\sqrt{x}} = \lim_{x \to \infty} e^{\ln((\ln(x))^{1/\sqrt{x}}} \]
\[= \lim_{x \to \infty} e^{\lim_{x \to \infty} \frac{\ln((\ln(x))^{1/\sqrt{x}}}{1}} \]
\[= \exp \left[ \lim_{x \to \infty} \frac{\ln((\ln(x))^{1/\sqrt{x}}}{1} \right] \]
\[= \exp \left[ \lim_{x \to \infty} \frac{\ln(\ln(x))}{1} \right] \]
\[= \exp \left[ \lim_{x \to \infty} \frac{1}{x \ln(x)} \right] = \exp [0] = 1 \]
Problem 7

\[ f'(x) = \frac{1}{3\sqrt{x}} - \sin x \]

\[ f'(x) = x^{\frac{1}{3}} - \sin x \]

\[ f(x) = \frac{x^{\frac{2}{3}}}{2} + \cos x + C \]

\[ f(x) = \frac{3}{2} x^{\frac{2}{3}} + \cos x + C \]

Extra Credit +6

\[ (fg)' = f' \cdot g + f \cdot g' + 1 \]

\[ f \text{ inc } \Rightarrow f' > 0. \]

Along with \( g < 0 \) \( \Rightarrow f'g < 0. \)

\[ g \text{ inc } \Rightarrow g' > 0. \]

Along with \( f < 0 \) \( \Rightarrow f'g < 0. \)

\[ f'g + f \cdot g' = \text{neg} + \text{neg} < 0. \]

\[ \Rightarrow (fg)' < 0 \text{ on I}. \]

\[ \Rightarrow fg \text{ decreases on I}. \]