**Composition of Functions**

**Definition:** The composition of two functions \( f \) and \( g \), denoted \( f \circ g \), is given by \((f \circ g)(x) := f(g(x))\).

**Meaning of the notation:** We start with two functions \( f \) and \( g \) that have their own domains and ranges. The notation \( f(g(x)) \) means that \( g \) goes first and \( f \) goes second. That is,

1. \( g \) picks up \( x \)'s, and delivers them, as \( g(x) \)'s, to \( f \);
2. next, \( f \) picks up the \( g(x) \)'s and delivers them to range \( f \).

**Example:** Given \( g(x) = \sqrt{x} \) and \( f(x) = 3x + 1 \), we'll find \( f \circ g \).

It may help to write \( f(x) \) as \( f(\Box) = 3 \cdot \Box + 1 \).

Repeat them

\[
\begin{align*}
    g(x) &= \sqrt{x} \\
    f(\Box) &= 3 \cdot \Box + 1
\end{align*}
\]

Then

\[
(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = 3 \cdot \sqrt{x} + 1 = 3\sqrt{x} + 1.
\]

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Example: Given \( g(x) = 4 + x^2 \) and \( f(x) = \frac{1}{5 \cdot x - 3} \), find \( f \circ g \).

Then

\[
(f \circ g)(x) = f(g(x)) = f(4 + x^2) = \frac{1}{5 \cdot (4 + x^2) - 3}
\]

\[
= \frac{1}{20 + 5x^2 - 3} = \frac{1}{17 + 5x^2}
\]

Assumption: To find the domain means to find the maximum domain.

Finding the domain of \( f \circ g \):

Since \( g \) goes first, the domain of \( f \circ g \) is limited by what \( g \) knows how to map, and \( g \) knows how to map only the numbers in domain \( g \). Therefore, domain \( f \circ g \) is limited by domain \( g \). In other words, domain \( f \circ g \) \( \subseteq \) domain \( g \).

Example: Given \( g(x) = \sqrt{x} \) and \( f(x) = x + 1 \), we can show that domain \( g \) limits domain \( f \circ g \).

Domain \( g = [0, \infty) \). Because \( g \) goes first in \( f \circ g \), \( f \circ g \) can use at most those numbers \( g \) can use.
Since \( g \) is limited to using numbers in \([0, \infty)\), \( f \circ g \) is limited to using numbers in \([0, \infty)\).

We have shown that \( \text{domain } f \circ g \subset \{0, \infty\} = \text{domain } g \).

**Find the domain of \( f \circ g \):**

Since \( f \) goes second in \( f \circ g \), \( f \) is dependent on what \( g \) gives to \( f \). The dependency means \( f \) receives only the \( g(x) \)'s that \( f \) knows how to use. Therefore, to find domain \( f \circ g \), we keep only the \( x \)'s in domain \( g \) that result in \( g(x) \)'s \( f \) can use — we keep only the \( x \)'s that result in \( g(x) \)'s landing inside domain \( f \).

**Definition:**
\[
\text{domain } f \circ g = \{ x \in \text{domain } g \mid g(x) \in \text{domain } f \}. 
\]

**Diagram:**

- **Domain of \( g \):**
  - Used numbers in \( \text{domain } g \)
  - Unused numbers in \( \text{domain } g \)

- **Domain of \( f \):**
  - Not used
  - Used

- **Range of \( g \):**
  - Not used
  - Used

- **Range of \( f \circ g \):**
  - Not used
  - Used

**Example:** Find domain \( f \circ g \) when \( f(x) = \sqrt{x} \) and \( g(x) = x^3 + 1 \).

Domain \( g = (-\infty, \infty) \), and domain \( f = [0, \infty) \).

Using the above definition for domain \( f \circ g \), we see that...
To find all x's in domain fog we look for all x's in domain that result in g(x)'s in domain f. That is, we want all x's in \( \text{dom } g = (-\infty, \infty) \) that result in g(x)'s in \( \text{dom } f = [0, \infty) \).

It is the x's in \([-1, \infty)\) that wind up as g(x)'s in dom f.
An interpretation of Definition A.

Def A: \( \text{domain } f \circ g = \{ x \in \text{domain } g \mid g(x) \in \text{domain } f \} \)

Result B: \( \text{domain } f \circ g = \text{dom } g \cap \{ x' \in \mathbb{R} \text{ such that } \exists x \in \text{dom } f \mid g(x) = x' \} \)

Example Find domain \( f \circ g \) when \( f(x) = \sqrt{x} \) and \( g(x) = x^2 + 1 \).

\[
\text{dom } g \cap \left\{ x' \in \mathbb{R} \mid g(x) = x' \right\} = (-\infty, \infty) \cap \{ x \in \mathbb{R} \mid x^2 + 1 \geq 0 \}
\]

\[
= (-\infty, \infty) \cap \{ x \in \mathbb{R} \mid x^2 \geq -1 \}
\]

\[
= (-\infty, \infty) \cap \{ x \in \mathbb{R} \mid x \geq -1 \}
\]

\[
= [-1, \infty)
\]

Example Find domain \( f \circ g \) when \( f(x) = \sqrt{x} \) and \( g(x) = \frac{3}{\sqrt{x}} \).

\[
\text{dom } f \circ g = \text{dom } g \cap \{ x \in \mathbb{R} \mid g(x) > 0 \}
\]

\[
= (-\infty, \infty) \cap \{ x \in \mathbb{R} \mid \frac{3}{\sqrt{x}} > 0 \}
\]
\[
\begin{align*}
&= (-\infty, \infty) \cap \{x \in \mathbb{R} \mid x \geq 0\} \\
&= (-\infty, \infty) \cap [0, \infty) \\
&= [0, \infty)
\end{align*}
\]

**Example**: Find \( \text{dom } f \circ g \) when \( f(x) = \frac{1}{1-x^2} \) and \( g(x) = \sqrt{x} + 3 \).

\[
\text{dom } f \circ g = \text{dom } g \cap \{x \in \mathbb{R} \mid g(x) \neq \pm 1\}
\]
\[
= [0, \infty) \cap \{x \in \mathbb{R} \mid \sqrt{x} - 3 \neq \pm 1\}
\]
\[
= [0, \infty) \cap \{x \in \mathbb{R} \mid \sqrt{x} \neq 4, 2\}
\]
\[
= [0, \infty) \cap [(-\infty, 4) \cup (4, 16) \cup (16, \infty)]
\]
\[
= [0, 4) \cup (4, 16) \cup (16, \infty)
\]
Notice: there is another way to find \( \text{dom} \, f \circ g \) in the above example.

\[
(f \circ g)(x) = f(g(x)) = f(\sqrt{x^2 - 3}) = \frac{1}{1 - (\sqrt{x^2 - 3})^2} = \frac{1}{1 - (x^2 - 3)^{\frac{1}{2}}}
\]

The final form of \((f \circ g)(x)\) is \(\frac{1}{1 - (\sqrt{x^2 - 3})^2}\) and its domain is \([0, \infty) \cap \{x \neq 4, 16\} = [0, 4) \cup (4, 16) \cup (16, \infty)\).
That is, \( \text{dom} [f \circ g] = \text{domain} [g \circ f \text{ of the final form of } f(g(x))] \).

**Definition.** Let the final expression of \( f(g(x)) \) be called the final form of \( (f \circ g)(x) \).

**Discussion.** in general, \( \text{dom} [f \circ g] \neq \text{dom} [\text{final form of } f(g(x))] \).

**Example.** Find \( \text{dom} [f \circ g] \) when \( f(x) = x^2 \) and \( g(x) = \sqrt{x} \).

\[
\text{dom} \ g = [0, \infty).
\]

The final form of \( f \circ g = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x \).

\[
\text{dom} [\text{final form}] = (-\infty, \infty).
\]

We see that \( \text{dom} [\text{final form}] \) is too large to be \( \text{dom} [f \circ g] \) because \( \text{dom} [\text{final form}] = (-\infty, \infty) \cap \text{dom} \ g = [0, \infty) \).

We look back to Definition 1 to find \( \text{dom} \ f \circ g \).

We want all \( x \) in \( \text{dom} \ g = [0, \infty) \) such that \( g(x) \) is in \( \text{dom} f = (-\infty, \infty) \).

All \( x \)'s in \( \text{dom} \ g \) satisfy \( g(x) \) in \( \text{dom} \ f \).

So \( \text{dom} (f \circ g) = [0, \infty) = \text{dom} \ g \).
**Result C:** \( \text{dom } (f \circ g) = \text{dom } g \cap \text{dom } \left[ \text{final form} \right] \)

**Example:** Find \( \text{dom } f \circ g \) when \( f(x) = \frac{1}{x-3} \) and \( g(x) = \sqrt{x} \).

Domain \( g = [0, \infty) \).

\((f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = \frac{1}{\sqrt{x} - 3} \)

Domain \( \left[ \text{final form} \right] = [0, \infty) \cap \{ x \neq 9 \} = [0, 9) \cup (9, \infty) \).

It turns out that

\[ \text{dom } [f \circ g] = \text{dom } g \cap \text{dom } \left[ \text{final form} \right] \]
\[ = [0, \infty) \cap (0, 9) \cup (9, \infty) \]
\[ = [0, 9) \cup (9, \infty) \]

We can verify that the answer is correct if we apply Definition A.

We want all \( x \) in \( \text{dom } g = [0, \infty) \) that result in \( g(x) \)'s landing in \( \text{dom } f = (-\infty, 3) \cup (3, \infty) \).

The only number in \( \text{dom } g \) not allowable in \( x = 9 \) because \( g(9) = 3 \) which is not in \( \text{dom } f \).

So \( \text{dom } [f \circ g] = [0, 9) \cup (9, \infty) \), once again.
Composition of Functions

\[ \text{dom} \ [f \circ g] \text{ winds up being all of dom}_g \text{ except } x = 9. \]
Proof of Result C. We must show \( \text{dom }[f \circ g] = \text{dom }[\text{final form}] \cap \text{dom } g \).

\( \Rightarrow \) Let \( x \) be in \( \text{dom }[f \circ g] \). By Definition A, 
\[ x \text{ is in } \text{dom } g \text{ such that } g(x) \text{ is in } \text{dom } f. \]
It means \( f(g(x)) \) is defined. Then \( x \) lies in \( \text{dom }[\text{final form}] \).

\( \Leftarrow \) Let \( x \) be in \( \text{dom } g \) such that \( x \) is \( \text{dom }[\text{final form}] \).
Then \( f(g(x)) \) is defined \( \Rightarrow \) \( g(x) \) must be in \( \text{dom } f \)
\[ \Rightarrow x \text{ lies in } \text{dom }[f \circ g]. \]

Some examples

\[
\begin{align*}
&[0, \infty) \quad \sqrt{x} \\
\Rightarrow & \quad (0, \infty) \quad x^3 \\
\Rightarrow & \quad [0, \infty) \quad (f \circ g)(x) = x^{3/2} \\
& \text{max dom } = [0, \infty)
\end{align*}
\]

\[
\begin{align*}
&[0, \infty) \quad x^3 \\
\Rightarrow & \quad (-\infty, 0) \quad \sqrt{x} \\
\Rightarrow & \quad (0, \infty) \quad (f \circ g)(x) = x^{3/2} \\
& \text{max dom } = (0, \infty)
\end{align*}
\]

\[
\begin{align*}
&[-1, \infty) \quad \sqrt{x+1} \\
\Rightarrow & \quad [-\infty, 0] \quad x^2 \\
\Rightarrow & \quad [0, \infty) \quad (f \circ g)(x) = x+1 \\
& \text{max dom } [-1, \infty)
\end{align*}
\]

\[
\begin{align*}
&(-\infty, \infty) \quad h(x+4) \\
\Rightarrow & \quad (-\infty, \infty) \quad e^x \\
\Rightarrow & \quad (-\infty, \infty) \quad (f \circ g)(x) = x+4 \\
& \text{max dom } (-\infty, \infty)
\end{align*}
\]