Commutativity of Addition in a Vector Space Follows from Most of the Other Axioms

Suppose that $V$ satisfies all the axioms of a vector space except perhaps the commutativity of vector addition and the associativity of scalar multiplication. That is, suppose that $V$ is a nonempty set together with operations $\oplus : V \times V \to V$ and $\odot : \mathbb{R} \times V \to V$ such that

(Axiom 1) For each $v, w, z \in V$, $(v \oplus w) \oplus z = v \oplus (w \oplus z)$.

(Axiom 2) There exists an element $0_V \in V$ such that $0_V \oplus v = v$ for all $v \in V$.

(Axiom 3) For each $v \in V$, there exists an element $-v \in V$ such that $-v \oplus v = 0_V$.

(Axiom 4) For each $v, w \in V$ and $a \in \mathbb{R}$, $a \odot (v \oplus w) = (a \odot v) \oplus (a \odot w)$.

(Axiom 5) For each $v \in V$ and $a, b \in \mathbb{R}$, $(a + b) \oplus v = (a \odot v) \oplus (b \odot v)$.

(Axiom 6) For each $v \in V$, $1 \odot v = v$.

Axiom 2 says that $V$ has what is called a left additive identity, denoted by $0_V$, and Axiom 3 says that each vector $v$ has what is called a left additive inverse, denoted by $-v$. First we prove that the left identity is also a right identity and that a left additive inverse is also a right additive inverse. We use the fact that the associativity of addition holds for any finite number of elements of $V$, not only for three elements as it is required in Axiom 1. We also use the facts that the left additive identity $0_V$ is unique, and the left additive inverse of an element of $V$ is unique. The proofs of these facts do not require the commutativity of addition in $V$ or the associativity of scalar multiplication.
For each \( v \in V \),
\[
(v \oplus (-v)) \oplus (v \oplus (-v)) = v \oplus ((-v) \oplus v) \oplus (-v)
\]
\[
= v \oplus 0_V \oplus (-v)
\]
\[
= v \oplus (-v)
\]
Add \(-(v \oplus (-v))\) to the left of both sides to obtain \( 0_V \oplus (v \oplus (-v)) = 0_V \), and so
\[
v \oplus (-v) = 0_V.
\]
Thus \(-v\) is a two-sided inverse of \( v \). Moreover,
\[
v \oplus 0_V = v \oplus (-v \oplus v) = (v \oplus (-v)) \oplus v = 0_V \oplus v = v
\]
and thus the element \( 0_V \) is a two-sided identity, which can be shown to be unique. (Of course we could have supposed the existence of right inverses and a right identity, and similarly proved that they are two-sided.)

Now we will show that addition in \( V \) must be commutative. On the one hand, using
Axiom 4 and then Axiom 5, we have
\[
(a + b) \odot (v \oplus w) = (a + b) \odot v \oplus (a + b) \odot w = (a \odot v) \oplus (b \odot v) \oplus (a \odot w) \oplus (b \odot w).
\]
On the other hand, using Axiom 5 and then Axiom 4, we have
\[
(a + b) \odot (v \oplus w) = a \odot (v \oplus w) \oplus b \odot (v \oplus w) = (a \odot v) \oplus (a \odot w) \oplus (b \odot v) \oplus (b \odot w).
\]
Thus
\[
(a \odot v) \oplus (b \odot v) \oplus (a \odot w) \oplus (b \odot w) = (a \odot v) \oplus (a \odot w) \oplus (b \odot v) \oplus (b \odot w).
\]
Add the additive inverse of \( a \odot v \) to the left of both sides of the equation, and add the additive inverse of \( b \odot w \) to the right of both sides of the equation, to obtain
\[
(b \odot v) \oplus (a \odot w) = (a \odot w) \oplus (b \odot v).
\]
Let \( a = b = 1 \) and use Axiom 6 to conclude that \( v \oplus w = w \oplus v \). Therefore addition in \( V \) is commutative.