Dr. Nestler - Math 13

Commutativity of Addition in a Vector Space Follows from Most of the Other Axioms

Suppose that V satisfies all the axioms of a vector space except perhaps the commutativity of vector addition and the associativity of scalar multiplication. That is, suppose that V is a nonempty set together with operations of addition $\oplus$ and scalar multiplication $\odot$ such that

(Axiom 1) For each $v, w \in V$, $v \oplus w \in V$.

(Axiom 2) For each $v \in V$ and $c \in \mathbb{R}$, $c \odot v \in V$.

(Axiom 3) For each $v, w, z \in V$, $(v \oplus w) \oplus z = v \oplus (w \oplus z)$.

(Axiom 4) There exists an element $0_V \in V$ such that $v \oplus 0_V = v$ for all $v \in V$.

(Axiom 5) For each $v \in V$, there exists an element $-v \in V$ such that $v \oplus -v = 0_V$.

(Axiom 6) For each $v, w \in V$ and $a \in \mathbb{R}$, $a \odot (v \oplus w) = (a \odot v) \oplus (a \odot w)$.

(Axiom 7) For each $v \in V$ and $a, b \in \mathbb{R}$, $(a + b) \oplus v = (a \odot v) \oplus (b \odot v)$.

(Axiom 8) For each $v \in V$, $1 \odot v = v$.

Axiom 4 says that V has what is called a right additive identity, denoted by 0_V, and Axiom 5 says that each vector v has what is called a right additive inverse, denoted by $-v$. First we prove that the right identity is also a left identity and that a right additive inverse is also a left additive inverse. We use the fact that the associativity of addition holds for any finite number of elements of V, not only for three elements as it is required in Axiom 3. We also use the facts that the right additive identity 0_V is unique, and the right additive inverse of an element of V is unique. The proofs of these facts do not require the commutativity of addition in V or the associativity of scalar multiplication.

For each $v \in V$,

$$\begin{aligned}(-v \oplus v) \oplus (-v \oplus v) &= -v \oplus (v \oplus -v) \oplus v \\ &= -v \oplus 0_V \oplus v \\ &= (-v \oplus 0_V) \oplus v \\ &= -v \oplus v\end{aligned}$$

Add $-(-v \oplus v)$ to the right of both sides to obtain $(-v \oplus v) \oplus 0_V = 0_V$, and so $-v \oplus v = 0_V$. Thus $-v$ is a two-sided inverse of v. Moreover,

$$0_V \oplus v = (v \oplus -v) \oplus v = v \oplus (-v \oplus v) = v \oplus 0_V = v$$

and thus the element 0_V is a two-sided identity, which can be shown to be unique. (Of course we could have supposed the existence of left inverses and a left identity, and similarly proved that they are two-sided.)

Now we will show that addition in V must be commutative. On the one hand, using Axiom 6 and then Axiom 7, we have

$$(a+b) \odot (v \oplus w) = ((a+b) \odot v) \oplus ((a+b) \odot w) = (a \odot v) \oplus (b \odot v) \oplus (a \odot w) \oplus (b \odot w)$$

On the other hand, using Axiom 7 and then Axiom 6, we have

$$(a+b) \odot (v \oplus w) = (a \odot (v \oplus w)) \oplus (b \odot (v \oplus w)) = (a \odot v) \oplus (a \odot w) \oplus (b \odot v) \oplus (b \odot w)$$

Thus

$$(a \odot v) \oplus (b \odot v) \oplus (a \odot w) \oplus (b \odot w) = (a \odot v) \oplus (a \odot w) \oplus (b \odot v) \oplus (b \odot w)$$

Add the additive inverse of $a \odot v$ to the left of both sides of the equation, and add the additive inverse of $b \odot w$ to the right of both sides of the equation, to obtain

$$(b \odot v) \oplus (a \odot w) = (a \odot w) \oplus (b \odot v)$$

Let $a = b = 1$ and use Axiom 8 to conclude that $v \oplus w = w \oplus v$. Therefore addition in V is commutative.