Dr. Nestler - 4.6 - Rank and Nullity of a Matrix

Previously we defined subspaces of a vector space, and then developed ways to generate

subspaces, for example by:

and:

Today we define and study three vector spaces associated with any matrix.

Part 1: Row Space and Column Space

Let A be an $m \times n$ matrix, with $A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

The row vectors of A are the $\quad \times \quad$ vectors

The column vectors of A are the $\quad \times \quad$ vectors

We can use these sets of vectors to generate vector spaces.

Definition. The row space of A is the subspace of spanned by the row vectors of A, and the column space of A is the subspace of spanned by the column vectors of A.

Theorem. If a matrix A is row-equivalent to a matrix B, then the row space of A is the same as the row space of B.

Proof:

So row operations do not change the row space, although they may change the column space.

If B is in row echelon form, then its nonzero row vectors are linearly independent. (Why?)

Thus if A is row-equivalent to a matrix B in echelon form, then the nonzero row vectors of B form a basis for the row space of A.

Example 1: The matrix $A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}$ can be row-reduced to echelon form

$B = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So the first three row vectors (1, 3, 1, 3), (0, 1, 1, 0), (0, 0, 0, 1)

form a basis for the row space of A (as well as the row space of B).

This vector space is a subspace of:

Example 2: We will find a basis for the subspace of $\mathbb{R}^3$ spanned by the set

$S = \{(-1, 2, 5), (3, 0, 3), (5, 1, 8)\}$.

Begin by forming the matrix $A = \begin{bmatrix} -1 & 2 & 5 \\ 3 & 0 & 3 \\ 5 & 1 & 8 \end{bmatrix}$ using the elements of S as its

__________ vectors. A can be row-reduced to echelon form

$B = \begin{bmatrix} 1 & -2 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. So the two nonzero row vectors form a basis for span(S).

So dim span$(S) = 2$.

Now suppose we wish to determine (a basis for) the column space of A instead. One way to do this is to find a basis for the row space of A^T.

Example: Let A be the matrix from Example 1. Then

$A^T =$

This row-reduces to echelon form $C = \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 0 & 1 & 9 & -5 & -6 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

and so the three nonzero row vectors of C form a basis for the row space of C
$=$ the row space of A^T
$=$ the column space of A.

Another way to do this: Consider, in Example 1, matrix B, the echelon form of this matrix A. Notice that column vectors ________________ of B are linearly independent.

Fact: Although the independent column vectors of B do not necessarily form a basis for the column space of A, the corresponding columns of A are independent and thus do form a basis for the column space.

Thus, in Example 1, columns number _____________________ of matrix A form a basis for the column space of A.

Notice that in Example 1, the dimension of the row space of A is

and the dimension of the column space of A is

Theorem. The row space and column space of a matrix have the same dimension.

Proof: See p. 198.

Definition. The rank of a matrix is the dimension of the row (or column) space of the matrix,

denoted by rank(A).

Example: In Example 1, rank$(A) =$

Part 2: Null Space

Now we move on to define a third vector space associated with a matrix.

If A is an $m \times n$ matrix, then the vector equation $AX = \overline{0}$ has at least one solution:

in the vector space

Theorem. The set of all solutions X to the equation $AX = \overline{0}$ forms a subspace of

Proof: This was 4.3 #54 on previous homework.

Definition. This subspace is called the null space or kernel of the matrix A, denoted by

$N(A)$. It is also called the solution space of the system $AX = \overline{0}$.

Example 3: We will find the null space of $A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix}$

and also a basis for that space.

Since the matrix equation $AX = \overline{0}$ is homogeneous, we may row-reduce just A rather

than the augmented matrix $[A|\,\overline{0}]$. (Why?)

A has row echelon form $B = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ corresponding to the linear system

$$\begin{cases} x_1 + 2x_2 \quad\quad + 3x_4 = 0 \\ \quad\quad\quad\quad x_3 + \ \ x_4 = 0 \end{cases}$$

There are two parameters: given any x_2 and x_4, it must be that $x_3 =$

and $x_1 =$

So the null space consists of all vectors of the form

and so a basis for the null space is

Definition. The dimension of the null space of A is the nullity of A, denoted by nullity(A).

Example: In Example 3 above, nullity$(A) =$

Now in Example 3, columns 1 and 3 of B are linearly independent and detemine the rank of the matrix. The other two columns correspond to the two free variables (parameters), and therefore determine the nullity of the matrix. This is true in general:

Theorem. If A is an $m \times n$ matrix of rank r, then the dimension of the null space of A is $n - r$.

That is, $n =$ rank$(A) +$ nullity(A).

Proof: See p. 202.

Example: We find the rank and nullity of $A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix}$

and also find a basis for its column space.

Note that the size of matrix A is:

Row-reduce A to the matrix in reduced echelon form $B = \begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Thus we see that the rank of B (and hence of A) is:

Therefore by the theorem, the nullity of A is:

Also as before we note that columns of B are linearly independent, and thus a basis for the column space of A is:

The next theorem summarizes some of the above results.

Theorem. If matrix A is row-equivalent to matrix B, then they have the same row space, rank and nullity.

Theorem. Let A be an $n \times n$ matrix. Then A is invertible if and only if nullity$(A) = 0$.

Proof:

Corollary: Let A be an $n \times n$ matrix. Then A is invertible if and only if rank$(A) =$

Proof:

Invertible Matrix Theorem. Let A be $n \times n$. The following 10 statements are equivalent:

1)

2)

3)

4)

5)

6)

7)

8)

9)

10)