Dr. Nestler - Math 13 - Course Preview

In the first half of the course, we will study the problem of solving systems of linear equations. In previous courses you have considered small systems, such as two equations in two variables, and used the methods of elimination and substitution to investigate their solutions. For a particular system, regardless of size, rather than writing down the variables each time, we can simply remember to keep the columns and rows of coefficients aligned properly. Such a rectangular arrangement of objects is called a *matrix*. We will define multiplication of matrices such that, for example, the system below can be expressed as the multiplicative matrix equation that follows. See if you can determine how the rule of matrix multiplication is defined.

$$\begin{cases} w - 2x - 3y + 4z = 10 \\ 2w - 3x - 4y - 12z = -2 \\ -3w + 6x + 18y - 100z = 0 \end{cases}$$

$$\begin{bmatrix} 1 & -2 & -3 & 4 \\ 2 & -3 & -4 & -12 \\ -3 & 6 & 18 & -100 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \\ 0 \end{bmatrix}$$

We can refer to the multiplicative matrix equation as simply $AX = B$. We will investigate properties of the matrix A of coefficients that determine whether or not the corresponding system of equations has a solution, and if so, the number of solutions, and the exact form of any existing solutions. So we consider as our objects of study rectangular arrays of numbers, or matrices, which have many interesting properties that may be reinterpreted in terms of systems of linear equations.

As you know, when solving a system of linear equations, replacing an equation by the sum of that equation with a constant multiple of another equation (that is, a *linear combination* of equations) results in an equivalent system, one that has the same solutions (if any) as the original. Often in mathematics this notion of forming linear combinations is both meaningful and useful. For another example, a linear function $h(x) = ax + b$ is simply a linear combination of the constant function $f(x) = 1$ and the identity function $g(x) = x$. Linear combinations of function derivatives and of vectors in the plane and in space are also studied in courses such as Math 11 (Multivariable Calculus) and Math 15 (Ordinary Differential Equations).

In the second half of the course, we begin our real introduction to linear algebra, in which the main object of study is any set that contains all linear combinations of its objects. Such a set is called a *vector space*. Examples include:

the set of real numbers

the set of complex numbers

the set of n-vectors $(x_1, \ldots, x_n)$ where the x_i are real (or complex) numbers

the set of continuous functions defined on a fixed interval

the set of polynomials of degree less than or equal to a fixed positive integer

the set of matrices of a fixed size

the set of solutions to a homogeneous system of linear differential equations

Rather than prove results about these sets (spaces) individually, we will prove theorems holding for abstract vector spaces. This is the power of mathematics: we make abstract definitions and then prove a general result that applies to many specific examples.

After studying basic properties of vector spaces, we consider the natural maps from one vector space to another. These are functions which respect the linear nature of the spaces, that is, functions f from a vector space V to a vector space W such that $f(ax) = af(x)$ for all elements x of V and numbers a, and $f(x_1 + x_2) = f(x_1) + f(x_2)$ for all elements x_1 and x_2 of V. Such a function is called a *linear transformation*. See if you can use this definition to prove that if f is a linear transformation, then $f(ax_1 + bx_2) = af(x_1) + bf(x_2)$; that is, a linear transformation sends linear combinations to linear combinations.

Note that when I say we will prove theorems, I mean that I will prove many results in class, and you will prove others on your own for homework and on exams. Due to its inherent abstract nature, linear algebra is traditionally the first mathematics course in which the content includes not only the usual definitions, examples, theorems and applications of the material, but also various techniques of mathematical proof, and how to write your arguments clearly and correctly using both English words and mathematical notation.

Dr. Nestler - Math 13 - Definitions and Theorems

A theorem or proposition is an important mathematical result or statement that has a proof. A theorem has both hypotheses (assumptions) and a conclusion.

A mathematical result of lesser importance may be called a lemma. Often the proof of a theorem is broken up into proofs of several lemmas.

A proof is a mathematical argument demonstrating that a statement is true.

A corollary to a theorem is a result that is proved rather quickly using that theorem.

A definition is a statement that assigns meaning to a specific term.

A definition does not require a proof. It does not have hypotheses or a conclusion. It gives the meaning of a word or words. A definition is prescriptive, not merely descriptive. For example, the phrase “perfectly round, with no corners or straight edges” describes a circle, but this description is not sufficient, since it also describes other shapes such as ellipses. Here is the definition of a circle: A circle is the set of points in a plane that are a fixed distance from a fixed point in that plane.

Dr. Nestler - Math 13 - Some Set Notation, Mathematical Shorthand and Proof Strategies

Set Notation:

$a \in S$ means "a is an element of the set S." We may also read this as "a is in S."

$\{a \in S : a$ satisfies condition $P\}$ is the set of all objects a in the set S such that a satisfies condition P

$S \subseteq T$ means "The set S is a subset of the set T"
which means "Each element of S is also an element of T"

$S \cap T$ is "the intersection of sets S and T"
which is the set of all objects that are elements of both S and T:
$S \cap T = \{a : a \in S$ and $a \in T\}$

$S \cup T$ is "the union of sets S and T"
which is the set of all objects that are elements of either S or T (or both):
$S \cup T = \{a : a \in S$ or $a \in T\}$

Mathematical Shorthand:

$\forall\, a \in S$ means "for all elements a of S"
$\exists\, a \in S$ means "there exists an element a of S"
$\exists!\, a \in S$ means "there exists a unique element a of S"

Proof Strategies:

To show $S \subseteq T$, show that for each s in S, s is in T.

To show that S is a proper subset of T, first show that $S \subseteq T$ and then find some element of T that is not in S.

To show $S = T$, show that $S \subseteq T$ and $T \subseteq S$.

To show $a \in S \cap T$, show that $a \in S$ and $a \in T$.

To show $a \in S \cup T$, show that $a \in S$ or $a \in T$ (or both).

To show there exists a unique a satisfying some property, first show that there exists some a satisfying the property, and then show uniqueness by (1) supposing there exists b satisfying the property and proving that $a = b$, or (2) supposing there exists $b \neq a$ satisfying the property and obtaining a contradiction.

Dr. Nestler - Math 13 and Math 15 - Solution of two linear equations in two variables

Claim: Consider the system of two linear equations in two variables $\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases}$

with a_{12} and a_{22} not both zero. This system has a unique solution if and only if

$a_{11}a_{22} - a_{12}a_{21} \neq 0$.

Proof. This is a proof by cases. Let $\Delta = a_{11}a_{22} - a_{12}a_{21}$.

Case 1: Suppose $a_{12} = 0$ (and $a_{22} \neq 0$), so the first equation of the system is $a_{11}x = b_1$. Either

$a_{11} = 0$ or $a_{11} \neq 0$.

Case 1(a): Suppose $a_{11} = 0$, so $\Delta = 0$, and either

(1) $b_1 = 0$ and the first equation in the system is $0 = b_1$, in which case the second equation

results in infinitely many solutions of the form $\left(x, \frac{b_2 - a_{21}x}{a_{22}}\right)$, or

(2) $b_1 \neq 0$ and the first equation in the system is $0 = b_1$, in which case the system has no

solution.

Case 1(b): Suppose $a_{11} \neq 0$, so $x = \frac{b_1}{a_{11}}$, and so a unique solution exists, satisfying $y = \frac{b_2 - \frac{a_{21}b_1}{a_{11}}}{a_{22}}$

Case 2: The case $a_{22} = 0$ (and $a_{12} \neq 0$) proceeds similarly, with the roles of the equations

interchanged.

Case 3: Suppose both $a_{12} \neq 0$ and $a_{22} \neq 0$. Then the system may be rewritten as

$$\begin{cases} y = \frac{b_1}{a_{12}} - \frac{a_{11}}{a_{12}}x \\ y = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x \end{cases}$$

The system has a unique solution if and only if the slopes of these lines are unequal:

$$-\frac{a_{11}}{a_{12}} \neq -\frac{a_{21}}{a_{22}} \Leftrightarrow -\frac{a_{11}}{a_{12}} + \frac{a_{21}}{a_{22}} \neq 0 \Leftrightarrow \frac{a_{11}}{a_{12}} - \frac{a_{21}}{a_{22}} \neq 0 \Leftrightarrow \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{12}a_{22}} \neq 0$$

Thus there is a unique solution if and only if $\Delta \neq 0$, as desired. $\square$

Corollary: The homogeneous system of equations $\begin{cases} a_{11}x + a_{12}y = 0 \\ a_{21}x + a_{22}y = 0 \end{cases}$

has a nontrivial solution if and only if $a_{11}a_{22} - a_{12}a_{21} = 0$.

Proof: A system of two linear equations in two variables has either no solution, a unique solution, or infinitely many solutions. Since a homogeneous system always has the trivial solution, the result follows from the theorem. $\square$

Dr. Nestler - Math 13 - 1.2 - Systems of m Linear Equations in n Unknowns

Previously we solved this system of 2 linear equations in 2 variables

$$\begin{cases} 2x + y = 5 \\ -4x + 6y = 12 \end{cases}$$

Now we will demonstrate a new method that has several advantages over simple substitution and elimination.

Definition. A matrix is a rectangular arrangement of numbers, called the entries or components of the matrix.

$$\begin{cases} 2x + y = 5 \\ -4x + 6y = 12 \end{cases} \qquad \begin{bmatrix} 2 & 1 & 5 \\ -4 & 6 & 12 \end{bmatrix}$$

$$\begin{cases} x + \frac{1}{2}y = \frac{5}{2} \\ -4x + 6y = 12 \end{cases} \qquad \begin{bmatrix} 1 & \frac{1}{2} & \frac{5}{2} \\ -4 & 6 & 12 \end{bmatrix}$$

$$\begin{cases} x + \frac{1}{2}y = \frac{5}{2} \\ 8y = 22 \end{cases} \qquad \begin{bmatrix} 1 & \frac{1}{2} & \frac{5}{2} \\ 0 & 8 & 22 \end{bmatrix}$$

$$\begin{cases} x + \frac{1}{2}y = \frac{5}{2} \\ y = \frac{11}{4} \end{cases} \qquad \begin{bmatrix} 1 & \frac{1}{2} & \frac{5}{2} \\ 0 & 1 & \frac{11}{4} \end{bmatrix}$$

There are three kinds of legitimate manipulation of rows of an augmented matrix that are called (elementary) row operations:

1) interchange 2 rows ("transposition")
2) multiply a row by a nonzero constant
3) replace a row with the sum of that row and a multiple of another row ("replacement")

These operations are legitimate because the resulting matrices correspond to equivalent systems, meaning that they have the same set of solutions (which may be empty). Subtraction of rows is not an elementary row operation; instead, adding a negative multiple of one row to another is an example of the allowable replacement operation.

The first nonzero entry of a row is called a pivot. For simplicity, we call the entry in the mth row and the nth column of a matrix the (m, n) entry. Gaussian elimination is the process of performing elementary row operations in the following prescribed order:

Consider the (1,1) entry. If it is 0, then interchange the first row with another row. If every entry in the first column is 0, then consider the first column with a nonzero entry, and make sure that the top entry of that column is nonzero by interchanging rows if necessary.

Make this pivot, usually the (1,1) entry, into a 1, either by multiplying that row by its reciprocal, or interchanging this row with another that has a 1 in the desired entry. (Choosing which rows to interchange will be your only freedom allowed in this process.)

Use the 1 to get 0's below that 1, from top to bottom, in order. Do this by using the replacement row operation, once for each row as required.

When you are done with the first column (or the first one with a nonzero entry), consider the next nonzero column to the right. Usually, this means you will make the (2,2) entry a 1 using row operations 1 and 2 as above.

Use the 1 to get 0's below, as before.

Continue, moving left to right, completing one column at a time.

Put any rows with all 0's at the bottom.

When you are done, the matrix is in (row-)echelon form. See the definition on p. 15.

Example: Solve the system $\begin{cases} x - y + z = 8 \\ 2x + 3y - z = -2 \\ 3x - 2y - 9z = 9 \end{cases}$ using Gaussian elimination.

Some advantages of using row reduction and matrices to solve linear systems:

1) The process is an algorithm that can be programmed on a computer.

2) It omits the redundant writing of variables and equal signs.

3) It works on linear systems of any size.

Example: Solve the system $\begin{cases} 2x + y = 1 \\ 4x + 2y = 6 \end{cases}$ using Gaussian elimination.

Another related method is Gauss-Jordan elimination:

Use the pivot 1 to get 0's **both above and below**, working within a column from top to bottom in order. (This is different from the method shown by our textbook author, who first obtains a row-echelon form, and then obtains 0's above the pivot 1's.) This puts the matrix in reduced echelon form. The reduced echelon form of a matrix is unique.

Example: Solve the system $\begin{cases} x_1 + 3x_2 - 5x_3 + x_4 = 4 \\ 2x_1 + 5x_2 - 2x_3 + 4x_4 = 6 \end{cases}$ using Gauss-Jordan elimination.

Definition. A system of m linear equations in n variables $\begin{cases} a_{11}x_1 + \cdots \ + a_{1n}x_n = b_1 \\ a_{21}x_1 + \cdots \ + a_{2n}x_n = b_2 \\ \quad \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$

is called an $m \times n$ system. The system is homogeneous if $b_i = 0$ for all $i = 1, \ldots m$.

Note that $x_1 = \cdots = x_n = 0$ is always a solution to a homogeneous system, called the zero solution or trivial solution. A homogeneous system therefore has either a unique solution or infinitely many solutions.

Theorem. If a homogeneous system of m equations in n variables has $n > m$, then there are infinitely many solutions.

Proof: Suppose there exist only finitely many solutions to an $m \times n$ homogeneous system. Then there exists only the trivial solution. So Gauss-Jordan elimination leads to the system

$$\begin{cases} x_1 \qquad\quad = 0 \\ \quad x_2 \qquad = 0 \\ \qquad \ddots \\ \qquad\quad x_n = 0 \end{cases}$$

and possibly extra equations of the form $0 = 0$. So $m \geq n$. So if the system has only finitely many solutions, then $m \geq n$. By the contrapositive, if $n > m$, then the system has infinitely many solutions.

Dr. Nestler - Math 13 - Definition of Vector Space

Definitions. Let V be a nonempty set on which two operations, called addition and scalar multiplication, are defined. If u and v are in V, the sum of u and v is denoted by $u \oplus v$, and if c is a real number, the scalar multiple of v by c is denoted by $c \odot v$. If the following ten axioms are satisfied, then V together with its operations is called a vector space, and its elements are called vectors:

1) For all u and v in V, $u \oplus v$ is in V.
2) For all u, v and w in V, $u \oplus (v \oplus w) = (u \oplus v) \oplus w$.
3) For all u and v in V, $u \oplus v = v \oplus u$.
4) There exists an element 0_V of V such that $v \oplus 0_V = v$ for all v in V.
5) For each v in V, there exists an element w in V such that $v \oplus w = 0_V$.
   We write $w = -v$.
6) For all c in $\mathbb{R}$ and v in V, $c \odot v$ is in V.
7) For all c and d in $\mathbb{R}$ and v in V, $c \odot (d \odot v) = (cd) \odot v$.
8) For all c and d in $\mathbb{R}$ and v in V, $(c + d) \odot v = (c \odot v) \oplus (d \odot v)$.
9) For all c in $\mathbb{R}$ and v and w in V, $c \odot (v \oplus w) = (c \odot v) \oplus (c \odot w)$.
10) For all v in V, $1 \odot v = v$.

Notes: We will prove that the vector 0_V is unique, and given an arbitrary vector v in V, the vector $-v$ is unique. We call 0_V the zero vector of V, and we call $-v$ the additive inverse of the vector v.

We often write $v + w$ for $v \oplus w$ and cv for $c \odot v$.

As a result of Axioms 1 and 6, we say that the set V is closed under its operations.
Axiom 2 says that vector addition is associative.
Axiom 3 says that vector addition is commutative.
Axiom 7 says that scalar multiplication is associative.
Axioms 8 and 9 say that scalar multiplication is distributive.

Informal definition: A vector space is a nonempty set closed under compatible operations with the usual properties of addition and scalar multiplication.

This is the definition of a real vector space, or a vector space over the reals, because all of the numbers c, called scalars, are real numbers. More generally, scalars can be chosen from any field, which is a set of numbers in which we can add, subtract, multiply and divide according to the usual laws of arithmetic. For example, if it is possible for scalars to be complex numbers, then the space is called a complex vector space.

Dr. Nestler - Math 13 - Determining whether a Set is a Vector Space

Suppose that you want to prove or disprove that a triple $(S, +, \cdot)$ is a vector space.

1. If you know that the set S is a subset of another (larger) set V, and that $V = (V, +, \cdot)$ is a vector space under the same operations, you could try to show that S is a subspace of V.

(a) If you can show that S contains the zero vector of V, then you know that S is nonempty, so go on to determine if S is closed under the vector space operations of V. If it is, then S is a subspace of V, and hence a vector space.

(b) If you can show that S does not contain the zero vector of V, then S is not a subspace of V, and hence not a vector space.

2. If you can't think of how to make S a subspace of a vector space, start verifying the 10 axioms from the definition of "vector space."

(a) Remember that the zero element of S would have to be the scalar multiple of any element of S with the number 0. Also, the additive inverse of an element s of S would have to be the scalar multiple of s with the number -1. Find these elements and check the related axioms and the two closure axioms first.

(b) If you think that S is not a vector space, show by a specific example that one of the axioms does not hold. For example, if you can find particular elements u, v and w such that $u + (v + w) \neq (u + v) + w$, then you are done.

Dr. Nestler - 4.6 - Rank and Nullity of a Matrix

Previously we defined subspaces of a vector space, and then developed ways to generate

subspaces, for example by:

and:

Today we define and study three vector spaces associated with any matrix.

Part 1: Row Space and Column Space

Let A be an $m \times n$ matrix, with $A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

The row vectors of A are the $\times$ vectors

The column vectors of A are the $\times$ vectors

We can use these sets of vectors to generate vector spaces.

Definition. The row space of A is the subspace of spanned by the row vectors of A, and the column space of A is the subspace of spanned by the column vectors of A.

Theorem. If a matrix A is row-equivalent to a matrix B, then the row space of A is the same as the row space of B.

Proof:

So row operations do not change the row space, although they may change the column space.

If B is in row echelon form, then its nonzero row vectors are linearly independent. (Why?)

Thus if A is row-equivalent to a matrix B in echelon form, then the nonzero row vectors of B form a basis for the row space of A.

<u>Example 1</u>: The matrix $A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}$ can be row-reduced to echelon form

$B = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So the first three row vectors (1, 3, 1, 3), (0, 1, 1, 0), (0, 0, 0, 1)

form a basis for the row space of A (as well as the row space of B).

This vector space is a subspace of:

<u>Example 2</u>: We will find a basis for the subspace of $\mathbb{R}^3$ spanned by the set

$S = \{(-1, 2, 5), (3, 0, 3), (5, 1, 8)\}$.

Begin by forming the matrix $A = \begin{bmatrix} -1 & 2 & 5 \\ 3 & 0 & 3 \\ 5 & 1 & 8 \end{bmatrix}$ using the elements of S as its

__________ vectors. A can be row-reduced to echelon form

$B = \begin{bmatrix} 1 & -2 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. So the two nonzero row vectors form a basis for span(S).

So dim span$(S) = 2$.

Now suppose we wish to determine (a basis for) the column space of A instead. One way to do this is to find a basis for the row space of A^T.

Example: Let A be the matrix from Example 1. Then

$A^T =$

This row-reduces to echelon form $C = \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 0 & 1 & 9 & -5 & -6 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

and so the three nonzero row vectors of C form a basis for the row space of C
$=$ the row space of A^T
$=$ the column space of A.

Another way to do this: Consider, in Example 1, matrix B, the echelon form of this matrix A. Notice that column vectors ________________ of B are linearly independent.

Fact: Although the independent column vectors of B do not necessarily form a basis for the column space of A, the corresponding columns of A are independent and thus do form a basis for the column space.

Thus, in Example 1, columns number ______________________ of matrix A form a basis for the column space of A.

Notice that in Example 1, the dimension of the row space of A is

and the dimension of the column space of A is

Theorem. The row space and column space of a matrix have the same dimension.

Proof: See p. 198.

Definition. The rank of a matrix is the dimension of the row (or column) space of the matrix,

denoted by rank(A).

Example: In Example 1, rank$(A) =$

Part 2: Null Space

Now we move on to define a third vector space associated with a matrix.

If A is an $m \times n$ matrix, then the vector equation $AX = \overline{0}$ has at least one solution:

in the vector space

Theorem. The set of all solutions X to the equation $AX = \overline{0}$ forms a subspace of

Proof: This was 4.3 #54 on previous homework.

Definition. This subspace is called the null space or kernel of the matrix A, denoted by

$N(A)$. It is also called the solution space of the system $AX = \overline{0}$.

Example 3: We will find the null space of $A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix}$

and also a basis for that space.

Since the matrix equation $AX = \overline{0}$ is homogeneous, we may row-reduce just A rather

than the augmented matrix $[A|\,\overline{0}]$. (Why?)

A has row echelon form $B = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ corresponding to the linear system

$$\begin{cases} x_1 + 2x_2 \qquad + 3x_4 = 0 \\ \qquad\qquad x_3 + \ \ x_4 = 0 \end{cases}$$

There are two parameters: given any x_2 and x_4, it must be that $x_3 =$

and $x_1 =$

So the null space consists of all vectors of the form

and so a basis for the null space is

Definition. The dimension of the null space of A is the nullity of A, denoted by nullity(A).

Example: In Example 3 above, nullity$(A) =$

Now in Example 3, columns 1 and 3 of B are linearly independent and detemine the rank of the matrix. The other two columns correspond to the two free variables (parameters), and therefore determine the nullity of the matrix. This is true in general:

Theorem. If A is an $m \times n$ matrix of rank r, then the dimension of the null space of A is $n - r$.

That is, $n =$ rank$(A) +$ nullity(A).

Proof: See p. 202.

Example: We find the rank and nullity of $A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix}$

and also find a basis for its column space.

Note that the size of matrix A is:

Row-reduce A to the matrix in reduced echelon form $B = \begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Thus we see that the rank of B (and hence of A) is:

Therefore by the theorem, the nullity of A is:

Also as before we note that columns of B are linearly independent, and thus a basis for the column space of A is:

The next theorem summarizes some of the above results.

Theorem. If matrix A is row-equivalent to matrix B, then they have the same row space, rank and nullity.

Theorem. Let A be an $n \times n$ matrix. Then A is invertible if and only if nullity$(A) = 0$.

Proof:

Corollary: Let A be an $n \times n$ matrix. Then A is invertible if and only if rank$(A) =$

Proof:

Invertible Matrix Theorem. Let A be $n \times n$. The following 10 statements are equivalent:

1)

2)

3)

4)

5)

6)

7)

8)

9)

10)