Dr. Nestler - Math 13 - Some results concerning dimension

One of your suggested homework problems (4.4 #70) is to prove this claim: Suppose $\{v_1, \ldots, v_n\}$ is a linearly independent set of vectors in a vector space V. If the set $\{v_1, \ldots, v_n, v\}$ is linearly dependent, then v is a linear combination of $v_1, \ldots, v_n$.

Contrapositive of 4.4 #70: Suppose $S = \{v_1, \ldots, v_n\}$ is a linearly independent set of vectors in a vector space V. If there exists a vector $v \notin \text{span}(S)$, then $\{v_1, \ldots, v_n, v\}$ is linearly independent.

On page 189 in section 4.5, you may read the proof of the following result.

Theorem 4.10: If V is an n-dimensional vector space, then any set of more than n vectors is linearly dependent.

We use the two results above to prove the following result.

Corollary (4.5 #83): If W is a subspace of a finite-dimensional vector space V, then W is finite-dimensional and dim $W \leq$ dim V.

Proof: This result is clear if V or W is $\{0_V\}$, so suppose otherwise. Let w_1 be a nonzero vector in W. If $W = \text{span}\{w_1\}$, then $\{w_1\}$ is a basis for W, and so W is finite-dimensional. Otherwise, there exists a nonzero vector $w_2 \in W$ such that $w_2 \notin \text{span}\{w_1\}$. Since $\{w_1\}$ is a linearly independent set, $\{w_1, w_2\}$ must also be linearly independent, by the contrapositive above.

If $W = \text{span}\{w_1, w_2\}$, then W has a basis of two vectors and so is finite-dimensional. If not, then there exists a nonzero vector $w_3 \in W$ such that $w_3 \notin \text{span}\{w_1, w_2\}$, and we continue as before. After each step, we have obtained a linearly independent set $\{w_1, \ldots, w_k\}$ in $W \subseteq V$.

Now if dim $V = n \geq 1$, then any set of more than n vectors of V must be linearly dependent, by the theorem above. Thus $k \leq n$. So $W = \text{span}\{w_1, \ldots, w_k\}$ for some $k \leq n$. Hence W is finite-dimensional and dim $W \leq$ dim V, as desired. $\square$

Using the contrapositive above, we may also prove the following result.

Theorem (4.5 #85): If S is a linearly independent set of vectors in a finite-dimensional vector space V, then there exists a basis for V containing S.

Proof: Because S is linearly independent, we know that S contains at least one vector that is not the zero vector, and hence dim $V > 0$. If S spans V, then S itself forms a basis for V and we are done. Assuming S does not span V, let v_1 be a vector in V that is not in span(S), and consider the set $S_1 = S \cup \{v_1\}$, which is linearly independent by the contrapositive above.

If S_1 spans V then we are done; if not, then there exists a vector $v_2 \notin S_1$, and we continue as before. Because V is finite-dimensional, this process must stop after a finite number of steps, ultimately producing a basis for V that contains S, as desired. $\square$

When we get to section 6.2, we will need the following result.

Claim: If W is an n-dimensional subspace of an n-dimensional vector space V, then $W = V$.

Proof: If $n = 0$, then this is clear, as both W and V are the trivial vector space $\{0_V\}$. Assuming $n \geq 1$, W has a basis B of n vectors. These n vectors are linearly independent in V, which has dimension n, so they span V. So each $v \in V$ is in span$(B) = W$, and thus $V \subseteq W$. Since clearly $W \subseteq V$, we have $W = V$, as desired. $\square$