Dr. Nestler - Math 13 - The Associativity of Matrix Multiplication

Here we prove that if A, B and C are matrices such that the products AB and BC are defined, then $A(BC) = (AB)C$. This is a direct proof that relies only on the definition of matrix multiplication, and not, in particular, on double sums or linear transformations of vector spaces. We begin with two important consequences of the definition of matrix multiplication.

(1) If B is a matrix and X is a column vector for which BX is defined, then the column vector BX is a linear combination of the columns of B, with coefficients being the entries of X. That is, if $B = [B_1|\cdots|B_n]$ with columns B_i, and $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, then $BX = x_1B_1 + \cdots + x_nB_n$. See page 46 of our textbook.

(2) If A and B are matrices such that AB is defined, then the ith column of AB is the product of A times the ith column of B; that is, $AB = A[B_1|\cdots|B_n] = [AB_1|\cdots|AB_n]$.

Now assume that the products AB and BX are defined as above. By (1) and the fact that matrix multiplication is a linear operation, we have

$$\begin{aligned} A(BX) = A(x_1B_1 + \cdots + x_nB_n) &= A(x_1B_1) + \cdots + A(x_nB_n) \\ &= x_1(AB_1) + \cdots + x_n(AB_n) \\ &= [AB_1|\cdots|AB_n]X = (AB)X \end{aligned}$$

where the final equality comes from (2). If we let X be an arbitrary column of matrix $C = [C_1|\cdots|C_p]$, where C is a matrix such that the product BC is defined, then we have $A(BC_i) = (AB)C_i$. Again, using (2), this means $A(BC) = (AB)C$, which proves that matrix multiplication is associative.