Dr. Nestler - Math 13 - Row and Column Spaces Have Equal Dimensions: Two More Proofs

Our textbook contains a proof, on pages 198-199, of the fact that the row space and column space of a matrix have the same dimension. Like most of the proofs of this fact appearing in introductory textbooks, it is sufficient but a bit off-putting due to its use of systems of equations and the significant amount of subscripts involved. Here we present two alternative proofs that are perhaps more conceptual and elegant.

We rely on an important fact explained on page 46: A product AX of a matrix A and a column vector X can be written as a linear combination of the columns of A with the entries of X as coefficients. That is, if $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [C_1 \,|\, C_2 \,|\, \cdots \,|\, C_n]$ where C_j is the j^{th} column of A, and $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ then $AX = x_1C_1 + x_2C_2 + \cdots + x_nC_n$.

Let row(A) denote the row space of A, and let col(A) denote the column space of A. We know that elementary row operations do not change the row space of a matrix. However, elementary column operations certainly can change the column space. For example, in exercise 41 on page 206, the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 2 & 5 & 1 & 1 & 0 \\ 3 & 7 & 2 & 2 & -2 \\ 4 & 9 & 3 & -1 & 4 \end{bmatrix}$ has reduced echelon form $B = \begin{bmatrix} 1 & 0 & 3 & 0 & -4 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Here col$(A) \neq$ col(B), since every vector in col(B) has its fourth component equal to 0 but this is clearly not true of col(A).

For our first proof, we will need a preliminary fact which is stated as exercise 80 in section 4.6.

Claim. If matrices A and B are row-equivalent, then the columns of A have the same dependence relationship as the columns of B.

Proof: Say that A has size $m \times n$ and X has size $n \times 1$ as above. Let $\overline{0}$ denote the $m \times 1$ zero vector. Suppose that a particular set of columns C_j of A are linearly dependent; therefore, there exists a nontrivial solution to the equation $AX = x_1C_1 + x_2C_2 + \cdots + x_nC_n = \overline{0}$ with the corresponding set of scalars x_j not all zero. Since A and B are row-equivalent, the solution sets of $AX = \overline{0}$ and $BX = \overline{0}$ are the same, and thus the corresponding columns of B are linearly dependent. $\square$

Theorem. The row and column spaces of an $m \times n$ matrix A have the same dimension.

Proof 1: Let R be the reduced row echelon form of A. By the claim above, row$(A) =$ row(R), and so dim(row$(A)) =$ dim(row(R))

$$= \text{number of nonzero rows of } R$$

$$= \text{number of leading ones of } R$$

Let this number be called r.

By the claim above, dim(col$(A)) =$ dim(col(R)). Since there are r leading ones, R has r columns that are standard unit vectors $e_1, \ldots, e_r$ of $\mathbb{R}^n$. These r vectors are linearly independent, and the remaining columns of R are linear combinations of them. Thus dim(col$(R)) = r$. Therefore dim(row$(A)) = r =$ dim(col A)), as desired. $\square$

In the second proof, gven a matrix A, let $R_i(A)$ denote the i^{th} row of A, and let $C_j(A)$ denote the j^{th} column of A.

Proof 2: Let $r = \dim(\text{row}(A))$, and let $c = \dim(\text{col}(A))$. Then there exists a set of c columns of A that are linearly independent and that span the column space of A. Let B be an $m \times c$ matrix having these columns. Each column of A is a linear combination of the c columns of B. That is, for $j = 1, \ldots, n$, the j^{th} column of A may be written as

$$C_j(A) = d_{1j}C_1(B) + d_{2j}C_2(B) + \cdots + d_{cj}C_c(B)$$

for some scalars d_{ij}. Define the matrix $D = [d_{ij}]$, which is of size $c \times n$, and so the matrix product BD is defined. Note that the right-hand side of the above equation is the same as the j^{th} column of BD. Thus

$$C_j(A) = C_j(BD)$$

Since A and BD have the same columns, they are equal: $A = BD$. So for $i = 1, \ldots, m$, we have

$$R_i(A) = R_i(BD) = b_{i1}R_1(D) + b_{i2}R_2(D) + \cdots + b_{ic}R_c(D)$$

So each row of A is a linear combination of the rows of D, and therefore $\dim(\text{row}(A))$ is no more than the number of rows of D; that is, $r \leq c$. The same argument may be applied to the transpose A^T, and so $\dim(\text{row}(A^T))$ is no more than $\dim(\text{col}(A^T))$. That is, $\dim(\text{col}(A))$ is no more than $\dim(\text{row}(A))$; in other words, $c \leq r$. Hence $r = c$, as desired. $\square$

Of course the common dimension of the row and columns spaces of A is called the rank of A. We say that $A = BD$ is a rank decomposition of the matrix A, meaning A is $m \times n$ with rank r, B is $m \times r$, and D is $r \times n$.