Dr. Nestler - Math 13 - Representing Elementary Row Operations

Stated without proof on p. 75 of our textbook is the fact that matrix multiplication on the left by an elementary matrix has the same effect as performing a corresponding elementary row operation on the original matrix. We prove this result here.

Theorem. Let A and B be $m \times n$ matrices. If B is obtained from A by an elementary row operation, then $B = EA$ where E is an $m \times m$ elementary matrix.

Proof: There are three cases, as there are three types of elementary row operations.

Case 1: Suppose B is obtained from A by interchanging rows p and q of A. Then let E be the elementary matrix obtained by interchanging rows p and q of the $m \times m$ identity matrix. Specifically, $E = [e_{ik}]$ is the $m \times m$ matrix such that

$$e_{ik} = \begin{cases} 1 & \text{if } i = k,\ i \neq p,\ i \neq q \\ 1 & \text{if } (i,k) = (p,q) \text{ or } (i,k) = (q,p) \\ 0 & \text{otherwise} \end{cases}$$

Let $[b_{ij}] = EA$, so by definition, $b_{ij} = \sum_{k=1}^{m} e_{ik} a_{kj}$.

If $i = p$, then $b_{ij} = b_{pj} = \sum_{k=1}^{m} e_{pk} a_{kj} = e_{pq} a_{qj} = a_{qj}$.

If $i = q$, then $b_{ij} = b_{qj} = \sum_{k=1}^{m} e_{qk} a_{kj} = e_{qp} a_{pj} = a_{pj}$.

If $i \neq p, q$ then $b_{ij} = \sum_{k=1}^{m} e_{ik} a_{kj} = e_{ii} a_{ij} = a_{ij}$.

In summary, $b_{ij} = \begin{cases} a_{ij} & \text{if } i \neq p, q \\ a_{qj} & \text{if } i = p \\ a_{pj} & \text{if } i = q \end{cases}$

Thus b_{ij} is the (i,j)th entry of the desired matrix B.

Case 2: Suppose B is obtained from A by multiplying row p of A by a nonzero scalar c. Then let E be the elementary matrix obtained by multiplying row p of the $m \times m$ identity matrix by c. Specifically, $E = [e_{ik}]$ is the $m \times m$ matrix such that

$$e_{ik} = \begin{cases} 1 & \text{if } i = k \neq p \\ c & \text{if } i = k = p \\ 0 & \text{if } i \neq k \end{cases}$$

Let $[b_{ij}] = EA$, so by definition, $b_{ij} = \sum_{k=1}^{m} e_{ik} a_{kj}$.

If $i = p$, then $b_{ij} = b_{pj} = \sum_{k=1}^{m} e_{pk} a_{kj} = e_{pp} a_{pj} = c a_{pj}$.

If $i \neq p$ then $b_{ij} = \sum_{k=1}^{m} e_{ik} a_{kj} = e_{ii} a_{ij} = a_{ij}$.

In summary, $b_{ij} = \begin{cases} a_{ij} & \text{if } i \neq p \\ c a_{ij} & \text{if } i = p \end{cases}$

Thus b_{ij} is the (i, j)th entry of the desired matrix B.

Case 3: Suppose B is obtained from A by adding c times row q to row p of A. Then let E be the elementary matrix obtained by adding c times row q to row p of the $m \times m$ identity matrix. Specifically, $E = [e_{ik}]$ is the $m \times m$ matrix such that

$$e_{ik} = \begin{cases} 1 & \text{if } i = k \\ c & \text{if } i = p \text{ and } k = q \\ 0 & \text{otherwise} \end{cases}$$

Let $[b_{ij}] = EA$, so by definition, $b_{ij} = \sum_{k=1}^{m} e_{ik} a_{kj}$.

If $i = p$, then $b_{ij} = b_{pj} = \sum_{k=1}^{m} e_{pk} a_{kj} = e_{pp} a_{pj} + e_{pq} a_{qj} = a_{pj} + c a_{qj}$.

If $i \neq p$ then $b_{ij} = \sum_{k=1}^{m} e_{ik} a_{kj} = e_{ii} a_{ij} = a_{ij}$.

In summary, $b_{ij} = \begin{cases} a_{ij} & \text{if } i \neq p \\ a_{pj} + ca_{qj} & \text{if } i = p \end{cases}$

Thus b_{ij} is the (i, j)th entry of the desired matrix B. $\square$

In fact, a similar argument can be used to prove the following result: if B is obtained from A by an elementary column operation, then $B = AE$ where E is an $n \times n$ elementary matrix.