Dr. Nestler - Math 13 - Are the Elementary Row Operations Really Elementary?

Typically, the elementary row operations on a matrix are defined to be of the following three types:

(1) Interchange two rows: $R_i \leftrightarrow R_j$

(2) Multiply a row by a nonzero scalar: $cR_i \to R_i$

(3) Add a scalar multiple of one row to another row: $cR_i + R_j \to R_j$

These operations are permissible because when we view a given matrix as the augmented matrix of a system of linear equations, performing any of these operations (or a sequence of them) results in an equivalent set of solutions to the system.

We might ask whether this is a minimal set of operations that has this property. In fact, this is not the case. For example, a row operation of type (1) can be achieved by a sequence of four row operations of types (2) and (3), as illustrated here on a 2 $\times$ 2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} a+c & b+d \\ -a & -b \end{bmatrix} \longrightarrow \begin{bmatrix} a+c & b+d \\ a & b \end{bmatrix} \longrightarrow \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

The operations, in order, are $R_2 + R_1 \to R_1$, $-R_1 + R_2 \to R_2$, $-R_2 \to R_2$, and $-R_2 + R_1 \to R_1$. This achieves the desired effect of $R_1 \leftrightarrow R_2$. This example immediately generalizes to the case of any two rows in a matrix of arbitrary size.

Mathematicians use this standard set of three elementary row operations because they are convenient for theory and computational practice. However, we might next ask if it is possible to streamline further the set consisting of row operations of type (2) and (3). In other words, could we have an even simpler set of row operations, sequences of which will produce the effects of all three standard elementary row operations? The answer is Yes. We could have just the two types

(2) Multiply a row by a nonzero scalar: $cR_i \to R_i$

(3') Add one row to another row: $R_i + R_j \to R_j$

This is because in order to add c times row i to row j, where $c \neq 0$, we can first multiply row i by c, then add the new row i to row j, and finally multiply row i by $\frac{1}{c}$.