Let $f(x, y)$ be a continuous function of two variables with continuous second partial derivatives such that $f_{xx}(a, b) \neq 0$, and let $\hat{u} = (u_1, u_2)$ be a unit vector. The directional derivative $D_{\hat{u}} f = f_x u_1 + f_y u_2$ is also a function of $x$ and $y$, so we can write its directional derivative in the direction of $\hat{u}$ as

$$D_{\hat{u}}^2 f = D_{\hat{u}} (D_{\hat{u}} f) = D_{\hat{u}} (f_x u_1 + f_y u_2)$$

$$= (f_x u_1 + f_y u_2)_x u_1 + (f_x u_1 + f_y u_2)_y u_2$$

$$= (f_{xx} u_1 + f_{yx} u_2) u_1 + (f_{xy} u_1 + f_{yy} u_2) u_2$$

$$= f_{xx} u_1^2 + 2 f_{xy} u_1 u_2 + f_{yy} u_2^2$$

Complete the square:

$$D_{\hat{u}}^2 f = f_{xx} \left( u_1^2 + \frac{2 f_{xy}}{f_{xx}} u_1 u_2 + \frac{f_{yy}}{f_{xx}} u_2^2 \right) + f_{yy} u_2^2 - \frac{f_{xx}^2}{f_{xx}^2} u_2^2$$

$$= f_{xx} \left( u_1 + \frac{f_{xy}}{f_{xx}} u_2 \right)^2 + \frac{u_2^2}{f_{xx}} \left( f_{xx} f_{yy} - f_{xy}^2 \right)$$

$$= f_{xx} \left( u_1 + \frac{f_{xy}}{f_{xx}} u_2 \right)^2 + \frac{u_2^2}{f_{xx}} D$$

If the discriminant $D$ and partial derivative $f_{xx}$ are positive at $(a, b)$, then the second derivative of $f$ in the direction of $\hat{u}$, $D_{\hat{u}}^2 f(a, b)$, is also positive, which implies that the vertical cross-section of the graph of $z = f(x, y)$ in the direction of $\hat{u}$ is concave up near $(a, b)$. So $f(a, b) \leq f(x, y)$ for all points $(x, y)$ sufficiently close to $(a, b)$, and hence $f(a, b)$ is a local minimum value.

Similarly, if $D > 0$ and $f_{xx} < 0$ at $(a, b)$, then $D_{\hat{u}}^2 f(a, b) < 0$, and so the graph is concave down near $(a, b)$, indicating $f$ has a local maximum value at $(a, b)$. \qed