Dr. Nestler - Math 11 - An Application of the Chain Rule: Implicit Partial Differentiation

Suppose we want to find $\frac{dy}{dx}$ where x and y are related implicitly by the equation $x^3 + y^3 = 6xy$. Rewrite this as $x^3 + y^3 - 6xy = 0$. Let $F(x, y) = x^3 + y^3 - 6xy$ so that the equation has the form $F(x, y) = 0$.

Since we are assuming that y is a differentiable function of x, we have $F(x, y(x)) = 0$ for all x in the domain of y. If $F(x)$ is differentiable, then the Chain Rule gives

$\frac{dF}{dx} = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$. If $\frac{\partial F}{\partial y} \neq 0$, then $\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$.

So in the example above, $\frac{dy}{dx} =$

Suppose an equation $F(x, y, z) = 0$ defines z implicitly as a differentiable function of independent variables x and y. (Sufficient conditions on the equation are given by an advanced result called the Implicit Function Theorem.) The Chain Rule gives

$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$; that is, $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$.

If $\frac{\partial F}{\partial z} \neq 0$, then $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ and similarly $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$.

Example: Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.