

Dr. Nestler - Math 11 - 15.4 Curvature (2nd lecture, handout)

Curvature measures how quickly a smooth, simple curve is bending.

We defined curvature by  $K = \left\| \frac{d\vec{T}}{ds} \right\|$ , the magnitude of the rate of change of the unit tangent vector with respect to arc length.

We proved that if a curve is given by a vector-valued function  $\vec{r}(t)$ , then

$$K = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

Example (15.5 #17): Show that the curvature at every point on the circular helix  $x = a \cos t$ ,

$y = a \sin t$ ,  $z = bt$  ( $a > 0$ ) is  $\frac{a}{a^2+b^2}$ . In particular, the curvature at all points is a constant.

$$\vec{r} = \langle a \cos t, a \sin t, bt \rangle$$

$$\vec{r}' = \langle -a \sin t, a \cos t, b \rangle$$

$$\vec{r}'' = \langle -a \cos t, -a \sin t, 0 \rangle$$

$$\begin{aligned} \vec{r}' \times \vec{r}'' &= \langle ab \sin t, -ab \cos t, a^2 \sin^2 t + a^2 \cos^2 t \rangle \\ &= \langle ab \sin t, -ab \cos t, a^2 \rangle \end{aligned}$$

$$\begin{aligned} \|\vec{r}' \times \vec{r}''\| &= \sqrt{a^2 b^2 (\sin^2 t + \cos^2 t) + a^4} \\ &= \sqrt{a^2 b^2 + a^4} = a \sqrt{a^2 + b^2} \end{aligned}$$

$$\|\vec{r}'\| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} = \sqrt{a^2 + b^2}$$

$$K = \frac{a \sqrt{a^2 + b^2}}{(\sqrt{a^2 + b^2})^3} = \frac{a}{a^2 + b^2}.$$

Notes: (1) It can be shown that lines and circles are the only plane curves with constant curvature. The example shows that this is not true for space curves.

(2) Consider different helices for different values of  $b$ .  $\lim_{b \rightarrow \infty} K = 0$ , which means the faster these curves climb, the smaller the curvature. Also, as  $b \rightarrow 0$ ,  $K \rightarrow \frac{1}{a}$ , which is the curvature of a circle of radius  $a$ .

Special case:  $C$  is a plane curve with parametric curves  $x = f(t)$ ,  $y = g(t)$ .

Then  $C$  is defined in  $\mathbb{R}^3$  by vector function  $\vec{r}(t) = \langle f(t), g(t), 0 \rangle$

$$\vec{r}'(t) = \langle f', g', 0 \rangle$$

$$\vec{r}''(t) = \langle f'', g'', 0 \rangle$$

$$\vec{r}' \times \vec{r}'' = \langle 0, 0, f'g'' - f''g' \rangle$$

$$K = \frac{|f'g'' - f''g'|}{(\sqrt{(f')^2 + (g')^2})^3} = \frac{|f'g'' - f''g'|}{((f')^2 + (g')^2)^{\frac{3}{2}}}.$$

To help remember this formula, notice that the numerator is the absolute value of this

$$\text{determinant: } \begin{vmatrix} f' & g' \\ f'' & g'' \end{vmatrix}$$

Example (15.4 #16): Find the curvature of curve  $x(t) = t - \sin t$ ,  $y(t) = 1 - \cos t$  at

$$P\left(\frac{\pi}{2} - 1, 1\right).$$

$P$  corresponds to the value  $t = \frac{\pi}{2}$ .

$$x'(t) = 1 - \cos t \quad y'(t) = \sin t$$

$$x''(t) = \sin t \quad y''(t) = \cos t$$

$$K\left(\frac{\pi}{2}\right) = \frac{|(\cos t)(1 - \cos t) - \sin^2 t|}{((1 - \cos t)^2 + \sin^2 t)^{\frac{3}{2}}} \Bigg|_{t=\frac{\pi}{2}} = \frac{1}{2^{\frac{3}{2}}} = \frac{\sqrt{2}}{4}.$$

An even more special case:  $C$  is a plane curve with equation  $y = g(x)$ .

Then  $\vec{r}(x) = \langle x, g(x), 0 \rangle$  with parameter  $x$ , so  $K = \frac{|g''(x)|}{(1 + (g'(x))^2)^{\frac{3}{2}}}.$

Example: Find the curvature of the parabola  $y = x^2$  at  $(0, 0)$ ,  $(1, 1)$  and  $(2, 4)$ .

$$y' = 2x, y'' = 2 \Rightarrow K(x) = \frac{2}{(1 + 4x^2)^{\frac{3}{2}}}.$$

The curvature at the origin is  $K(0) = 2$ , at  $(1, 1)$  is  $K(1) = \frac{2}{5^{\frac{3}{2}}} \approx 0.18$ ,

and at  $(2, 4)$  is  $K(2) = \frac{2}{17^{\frac{3}{2}}} \approx 0.03$ .

$\lim_{x \rightarrow \pm\infty} K(x) = 0$ : the parabola becomes flatter away from origin.

Example (15.4 #23): Find the points on the curve  $y = e^{-x}$  at which the curvature is a maximum.

$$y' = -e^{-x} \quad y'' = e^{-x}$$

$K(x) = \frac{|y''|}{(1+(y')^2)^{\frac{3}{2}}} = \frac{e^{-x}}{(1+e^{-2x})^{\frac{3}{2}}}$ . To maximize the function  $K(x)$ , we find its critical

points:

$$K'(x) = \frac{(1+e^{-2x})^{\frac{3}{2}}(-e^{-x}) - e^{-x}\frac{3}{2}(1+e^{-2x})^{\frac{1}{2}}(-2e^{-2x})}{(1+e^{-2x})^3} = 0$$

$$\frac{-e^{-x}(1+e^{-2x})+3e^{-3x}}{(1+e^{-2x})^{\frac{5}{2}}} = 0$$

$$\frac{-e^{-x}+2e^{-3x}}{(1+e^{-2x})^{\frac{5}{2}}} = 0$$

$$-e^{-x}(1 - 2e^{-2x}) = 0$$

$$2e^{-2x} = 1$$

$$e^{-2x} = \frac{1}{2}$$

$$2x = \ln \frac{1}{2} = \ln 2^{-1} = -\ln 2$$

So the only critical point is  $x = \frac{1}{2}\ln 2 \approx 0.35$ .

Using the 1st or 2nd derivative test, you can verify that  $K$  has a maximum at this critical point.

$$x = \frac{1}{2}\ln 2 = \ln \sqrt{2} \Rightarrow y = e^{-\ln \sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

So the curvature is a maximum at the point  $(\ln \sqrt{2}, \frac{\sqrt{2}}{2})$ .

Homework: 15.4 Read Example 6, can do #7-12, 23-32, 44-47

15.7 (Chapter 15 Review) can do #16, 17 19

You may wish to skim section 15.6 on Kepler's Laws. This is an incredible application of calculus and vectors, together with Newton's laws of motion, to obtain Kepler's three laws describing the motion of planets in our solar system.