Dr. Nestler - Math 10 - Pigeonhole Principle

Theorem (Pigeonhole Principle). Suppose $m$ and $n$ are positive integers such that $m > n$. If $f$ is a function from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$, then $f$ is not one-to-one.

Proof: This is a proof by induction on $n$. For the base case, suppose $n = 1$ and suppose $f$ is a function from $\{1, \ldots, m\}$ to $\{1\}$. Then $f(1) = f(m)$ and $m > 1$, so $f$ is not one-to-one.

Now suppose that the claim is true for some particular integer $n \geq 1$. (Normally we would use a different letter such as $k$, but soon we will be using several additional letters, and so this should be easier to follow.) So our inductive hypothesis is that if $m$ is an integer greater than $n$ and $f$ is a function from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$, then $f$ is not one-to-one.

We need to show that the claim is true for the integer $n + 1$. Suppose that $m$ is an integer greater than $n + 1$ and $f$ is a function from $\{1, \ldots, m\}$ to $\{1, \ldots, n + 1\}$. We will show that $f$ is not one-to-one. There are three distinct cases to consider: (1) either $f$ sends no integer to $n + 1$, or (2) $f$ sends more than one integer to $n + 1$, or (3) $f$ sends exactly one integer to $n + 1$.

Case 1: If $n + 1$ is not in the range of $f$, then in fact $f$ is a function from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$. Since $m > n + 1 > n$, the induction hypothesis implies that $f$ is not one-to-one, as desired.

Case 2: If there exist distinct integers $j, k \in \{1, \ldots, m\}$ such that $f(j) = f(k) = n + 1$, then clearly $f$ is not one-to-one, as desired.

Case 3: Suppose that $j \in \{1, \ldots, m\}$ is the only integer such that $f(j) = n + 1$. Define a new function $g : \{1, \ldots, m\} \to \{1, \ldots, m\}$ which interchanges $m$ with $j$ and fixes all other integers in its domain. That is,

$$g(x) = \begin{cases} x & \text{if } x \neq j, m \\ j & \text{if } x = m \\ m & \text{if } x = j \end{cases}$$

By definition, $g$ is one-to-one. Since $j$ is the only integer that $f$ sends to $n + 1$, $(f \circ g)(x) = f(g(x)) = n + 1$ if and only if $g(x) = j$. Since $g$ is one-to-one, this occurs if and only if $x = m$. Thus the composition $f \circ g$ sends $\{1, \ldots, m - 1\}$ to $\{1, \ldots, n\}$. Now $m > n + 1$, so $m - 1 > n$, so by the inductive hypothesis, $f \circ g$ is not one-to-one on the set $\{1, \ldots, m - 1\}$. Since $g$ is one-to-one, it follows that $f$ cannot be one-to-one, as desired.

We are done by the principle of mathematical induction. □

Corollary: The size of a nonempty finite set is well-defined. That is, if $A$ is a nonempty finite set, then there exists a unique positive integer $n$ such that $A \cong \{1, \ldots, n\}$.

Proof: Suppose that there exist distinct positive integers $m$ and $n$ such that $A \cong \{1, \ldots, m\}$ and $A \cong \{1, \ldots, n\}$. Without loss of generality, we may assume $m > n$. Since $\{1, \ldots, m\} \cong \{1, \ldots, n\}$, there exists a one-to-one function from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$. However this contradicts the Pigeonhole Principle. Hence $n$ is unique. □

Note: Exercise #52 in section 5.1 on mathematical induction asks for a proof of this theorem, although there is no mention that this is a formal statement of the Pigeonhole Principle. The author's proof implicitly relies on the result of the corollary, which is a consequence of Corollary 1 on page 400, which the author proves using exercise #52 in section 5.1....