Dr. Nestler - Math 10 - Some set proofs

Example (2.2 #6a): Prove that $A \cup \emptyset = A$.

$A \cup \emptyset = \{x : x \in A$ or $x \in \emptyset\} = \{x : x \in A\}$ since $\emptyset$ has no elements.

Example (2.2 #23): Prove the following distributive property:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

To show that these two sets are equal, we will show that each set is a subset of the other set.

If $x \in A \cup (B \cap C)$, then $x \in A$ or $x \in B \cap C$.

If $x \in A$ then $x \in A \cup B$ and $x \in A \cup C$, so $x \in (A \cup B) \cap (A \cup C)$.

If $x \in B \cap C$ then $x \in B \subseteq A \cup B$ and $x \in C \subseteq A \cup C$, so $x \in (A \cup B) \cap (A \cup C)$.

Thus $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

If $x \in (A \cup B) \cap (A \cup C)$, then $x \in A \cup B$ and $x \in A \cup C$, so $x \in A$ or $x \in B$,

and also $x \in A$ or $x \in C$.

Case 1: If $x \in A$ then $x \in A \cup (B \cap C)$.

Case 2: If $x \notin A$ then $x \in B$ and $x \in C$, so $x \in B \cap C \subseteq A \cup (B \cap C)$.

Thus $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Thus we have proved $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Example (2.2 #15): Prove the following De Morgan law: $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

If $x \in \overline{A \cup B}$, then $x \notin A \cup B$, so x is not in either A or B. So $x \notin A$ and $x \notin B$.

That is, $x \in \overline{A}$ and $x \in \overline{B}$. So $x \in \overline{A} \cap \overline{B}$, and thus $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$.

Now if $x \in \overline{A} \cap \overline{B}$, then $x \in \overline{A}$ and $x \in \overline{B}$. That is, x is not in A and x is not in B.

So x is not in $A \cup B$. So $x \in \overline{A \cup B}$, and thus $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$.

Thus $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Dr. Nestler - Math 10 - Supplemental Homework

Cardinality and countable sets:

1. Use a bijection to prove that $A = \{\dots, -7, -4, -1, 2, 5, 8, \dots\}$ is countable.

2. Prove that $\mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+ = \{(a, b, c) : a, b, c \in \mathbb{Z}^+\}$ is countable.

3. If S and T are countably infinite, prove that $S \times T$ is countably infinite.

4. Suppose that A is finite and B is countably infinite. Prove that $A \cup B$ is countable.

   Idea of Proof: $A \cup B = A \cup (B \backslash A)$, so we may assume A and B are nonempty and disjoint. There exist one-to-one correspondences $g : A \to \{1, \dots, n\}$ and $h : B \to \mathbb{Z}^+$.

   Define $f : A \cup B \to \mathbb{Z}^+$ by $f(x) = \begin{cases} g(x) & \text{if } x \in A \\ h(x) + n & \text{if } x \in B \end{cases}$

   Verify that f is injective. (In fact we can also verify that f is surjective.)

5. Prove that $A = \{\dots, -7, -4, -1, c, r, m, 2, 5, 8, \dots\}$ is countable.

6. Suppose that A and B are countable. Prove that $A \cup B$ is countable.

   Idea of Proof: We may assume A and B are disjoint and countably infinite, so there exists a bijection $g : A \to \mathbb{Z}^+$ and a bijection $h : B \to \mathbb{Z}^+$.

   Define $f : A \cup B \to \mathbb{Z}^+$ by $f(x) = \begin{cases} 2g(x) - 1 & \text{if } x \in A \\ 2h(x) & \text{if } x \in B \end{cases}$

   Verify that f is injective. (In fact we can also verify that f is surjective.)

Cardinality and countable and uncountable sets:

7. If A and B are countable sets, must $A - B$ be countable? Explain. (If yes, give a proof. If no, give a counterexample.)

8. If A and B are uncountable sets that are not disjoint, i.e. $A \cap B \neq \emptyset$, must $A \cap B$ be uncountable? Explain.

9. Determine a one-to-one correspondence between the open interval $(0, 1)$ and the open interval $(2, 7)$.

Dr. Nestler - Math 10 - Congruence and Modular Arithmetic

In exercises #1-20, perform the indicated calculations in $\mathbb{Z}_k$. Write your answer in the form $[r]$ with $0 \leq r < k$.

1. $[8]+[6]$ in $\mathbb{Z}_{12}$
2. $[9]+[11]$ in $\mathbb{Z}_{15}$
3. $[5]+[10]$ in $\mathbb{Z}_{11}$
4. $[9]+[8]$ in $\mathbb{Z}_{13}$
5. $[23]+[15]$ in $\mathbb{Z}_{8}$
6. $[12]+[25]$ in $\mathbb{Z}_{7}$
7. $[16]+[9]$ in $\mathbb{Z}_{6}$
8. $[43]+[31]$ in $\mathbb{Z}_{22}$
9. $[8][7]$ in $\mathbb{Z}_{6}$
10. $[9][3]$ in $\mathbb{Z}_{4}$
11. $[4][11]$ in $\mathbb{Z}_{9}$
12. $[3][20]$ in $\mathbb{Z}_{11}$
13. $[5][12]$ in $\mathbb{Z}_{8}$
14. $[8][11]$ in $\mathbb{Z}_{5}$
15. $[9][6]$ in $\mathbb{Z}_{10}$
16. $[16][3]$ in $\mathbb{Z}_{7}$
17. $[9]^7$ in $\mathbb{Z}_{7}$
18. $[11]^8$ in $\mathbb{Z}_{5}$
19. $[11]^9$ in $\mathbb{Z}_{12}$
20. $[13]^6$ in $\mathbb{Z}_{15}$

21. Must it be true that in $\mathbb{Z}_k$ if $[x][y]=[0]$ then $[x]=[0]$ or $[y]=[0]$? If yes, give a proof. If no, give a counterexample.

22. Use modular arithmetic to determine the last digit of 2^{30}.

23. Find $[2^{13}]$ in $\mathbb{Z}_{209}$. That is, find the integer r such that $0 \leq r < 209$ and $2^{13} \equiv r \pmod{209}$.

Dr. Nestler - Math 10 - Multinomial Coefficients: Counting and Supplemental Homework

Theorem. The multinomial coefficient $\binom{n}{i_1, \ldots, i_k}$ equals the number of ways to divide a set of size n into k mutually disjoint subsets $A_1, \ldots, A_k$ such that A_j has size i_j for each $j = 1, \ldots, k$.

Proof: In order to divide a set S of size n into k mutually disjoint subsets $A_1, \ldots, A_k$, where A_j has size i_j for each $j = 1, \ldots, k$, we perform a sequence of k tasks: choose i_1 elements of S to form the first subset A_1, then choose i_2 of the remaining $n - i_1$ elements of S to form the second subset A_2, and so on. The k^{th} and final task is to choose i_k elements of the remaining $n - i_1 - i_2 - \cdots - i_{k-1}$ elements of S to form the k^{th} subset A_k. (Since $n - i_1 - i_2 - \cdots - i_{k-1} = i_k$, there is only one way to perform the final task.)

By the Multiplication Rule, the number of ways to perform this sequence of tasks is

$$\binom{n}{i_1}\binom{n-i_1}{i_2}\binom{n-i_1-i_2}{i_3}\cdots\binom{n-i_1-i_2-\cdots-i_{k-1}}{i_k} = \frac{n!}{i_1!i_2!\cdots i_k!} = \binom{n}{i_1, \ldots, i_k}$$

Supplemental homework exercises:

1. Find the coefficient of x^5z^4 in the expansion of $(x + y + z + w)^9$.
2. Find the coefficient of $x_1^2x_2^2x_3^2x_4^2$ in the expansion of $(x_1 + x_2 + x_3 + x_4)^8$.
3. Find the number of ways to divide the set $S = \{a, b, c, d, e, f\}$ into three mutually disjoint subsets A_1 of size 3, A_2 of size 1, and A_3 of size 2.
4. Find the number of ways to divide the set $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ into five mutually disjoint subsets, each of size 2.
5. Find the number of ways to divide the set $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ into disjoint subsets A_1 of size 2, A_2 of size 2, A_3 of size 2, A_4 of size 3, and A_5 of size 1.
6. Find the number of ternary words of length 8 that contain exactly two 0's, three 1's and three 2's.

Dr. Nestler - Math 10 - Eulerian Circuits and Paths

Theorem. A connected graph with at least two vertices is Eulerian if and only if each of its vertices has even degree.

Proof: ($\Rightarrow$) Imagine tracing an Eulerian circuit, starting in the middle of an edge. Each time the circuit passes through a vertex, it contributes 2 to the degree of that vertex. Since each edge of the graph appears once in the circuit, each vertex has a degree that is a multiple of 2, and hence even.

($\Leftarrow$) Suppose every vertex of a connected graph G has even degree. We prove that G is Eulerian by induction on the number of edges in G. If this number is 1 or 2, then either

G is a loop 

or two loops 

or a circuit of length 2 

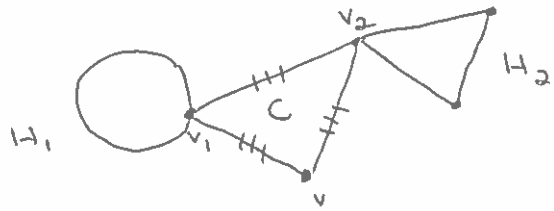
so the graph clearly contains an Eulerian circuit.

Assume that the result is true for all graphs with fewer than n edges. Suppose G has n edges, and let v be a vertex of G.

Claim: There exists a simple circuit C in G starting at v.

Proof of claim: Of all simple paths in G starting at v, let C be one with the greatest length, i.e. the most edges. If C does not end at v, then it ends at some other vertex w. But deg(w) is even, so the path must leave w. This contradicts C having the maximal length of the simple paths starting at v. Thus the path begins and ends at v, and is a circuit. □

If C contains all the edges of G, then it is an Eulerian circuit, and we are done. Otherwise, remove from G the edges of C and any vertices that would become isolated. The remaining subgraph H still has all vertices of even degree, but may be disconnected.



By the inductive hypothesis, each connected component of H contains an Eulerian circuit, and each of these circuits has a vertex on C. Start at v and travel along C until arriving at a vertex v_1 on an Eulerian circuit of a component H_1 of H. Complete this circuit, returning to v_1, and continue along C to a vertex v_2 that is on an Eulerian circuit of component H_2 of H, and so on. Since the graph is finite, this process constructs an Eulerian circuit in G. □

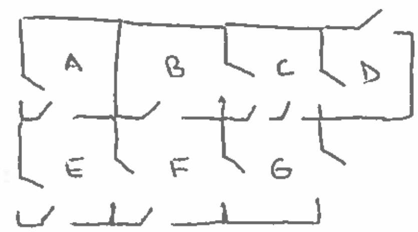
<u>Corollary</u>: A connected graph has an Eulerian path but not an Eulerian circuit if and only if it has exactly two vertices of odd degree.

<u>Proof</u>: ($\Leftarrow$) Suppose u and v are the only vertices of G of odd degree. Adding an edge $\{u, v\}$ results in a graph G_1 that is connected and with all vertices of even degree. Hence, by the theorem, G_1 has an Eulerian circuit C, and so removing $\{u, v\}$ from C results in an Eulerian path in G.

($\Rightarrow$) Suppose there exists an Eulerian path from u to v in G, but not an Eulerian circuit. The first edge of the path contributes 1 to deg(u), and a contribution of 2 is made each time the path passes through u. Symmetrically, the last edge in the path contributes 1 to deg(v), and a contribution of 2 is made each time the path passes through v. Therefore deg(u) and deg(v) are odd, and the degree of every other vertex is even because the path contributes 2 to the degree of a vertex when it passes through it. □

Dr. Nestler - Math 10 - Eulerian circuit exercises

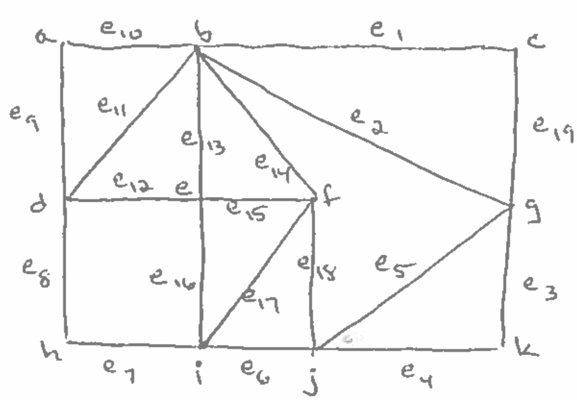
Example: Can we visit the seven rooms and the surrounding corridor using each door just once?



Fleury's Algorithm. To find an Eulerian circuit in an Eulerian graph:

1. Start at a vertex and follow edges, erasing them as they are used. Also erase any isolated vertices as they are created.
2. Do not choose an edge that is a bridge unless there is no other choice.

Example:



Dr. Nestler - Math 10 - An equivalent definition for trees

Lemma (10.4 #59): If there exist two simple paths between two vertices in a simple graph that do not contain the same set of edges, then the graph contains a simple circuit.

Proof: Suppose P_1 is a simple path $x_0, x_1, \ldots, x_n$ from $x_0 = u$ to $x_n = v$, and P_2 is a simple path $y_0, y_1, \ldots, y_m$ from $y_0 = u$ to $y_m = v$. The paths both begin at u, and since the sets of edges are not the same, they must diverge eventually. If this occurs only after one path has ended, then the rest of the other path is a simple circuit from v to v.

Otherwise we may suppose that $x_0 = y_0$, ..., $x_i = y_i$ but $x_{i+1} \neq y_{i+1}$. To create a simple circuit, follow the path y_i, y_{i+1}, and so on, until it once again encounters a vertex on P_1. Once we are back on P_1, we follow it along, forwards or backwards as necessary, to return to x_i. Since $x_i = y_i$, this forms a circuit.

It must be a simple circuit, since no edge among the x_k vertices or the y_k vertices can be repeated, as the paths P_1 and P_2 are simple, and also since no edge among the x_k vertices can be the same as one of the edges among the y_i vertices, since we left P_2 for P_1 as soon as we encountered P_1. □

Theorem. A graph is a tree if and only if there exists a unique simple path between each pair of vertices.

Proof: Read on p. 747.

Dr. Nestler - Math 10 - Euler's Formula

Theorem (Euler's Formula). If G is a connected, simple planar graph with v vertices, e edges and f faces, then $v - e + f = 2$.

Proof: A tree has just one face, so the formula holds for trees if and only if $v - e + 1 = 2$; that is, $e = v - 1$. In class, we prove that $e = v - 1$ for trees, and thus the formula holds for trees.

The formula is clearly true if $e = 0$ since in that case $v = f = 1$. Suppose that the formula holds for all graphs with at most n edges, and let G be a graph with $n + 1$ edges.

If G has no circuit, then G is a tree, and we know the formula holds for trees. Otherwise G has a circuit; remove an edge p on a circuit to obtain the subgraph G'. Since the circuit separates the plane into two regions, when we remove p, these regions merge, so G' has one fewer face than G. Since G' has n edges, we are supposing the formula holds for G': $v' - e' + f' = 2$. Now $v' = v$, $e' = e - 1$ and $f' = f - 1$, so $v - (e - 1) + (f - 1) = 2$; that is, $v - e + f = 2$. We are done by induction. $\square$

16.

Character	*Frequency*
U	11
T	10
S	4
Y	30
D	5

a) Find a Huffman code for the character set using the given frequencies.
b) Compute the weight of the code.
c) Decode the message 01000110001011.

17.

Character	*Frequency*
h	12
p	40
g	15
a	8
r	17

a) Find a Huffman code for the character set using the given frequencies.
b) Compute the weight of the code.
c) Encode the message "harp."
d) Decode the message 1101111000101.

18.

Character	*Frequency*
α	4
β	5
+	7
*	8
(	12
)	25

a) Find a Huffman code for the character set using the given frequencies.
b) Compute the weight of the code.
c) Encode the message "$\alpha(\alpha+\beta)$."
d) Decode the message 11001011101101111110010011010.

19.

Character	*Frequency*
h	13
p	7
y	9
d	22
a	24
*	27

a) Find a Huffman code for the character set using the given frequencies.
b) Compute the weight of the code.
c) Encode the message "(x + y)."