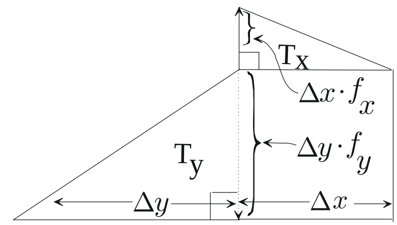
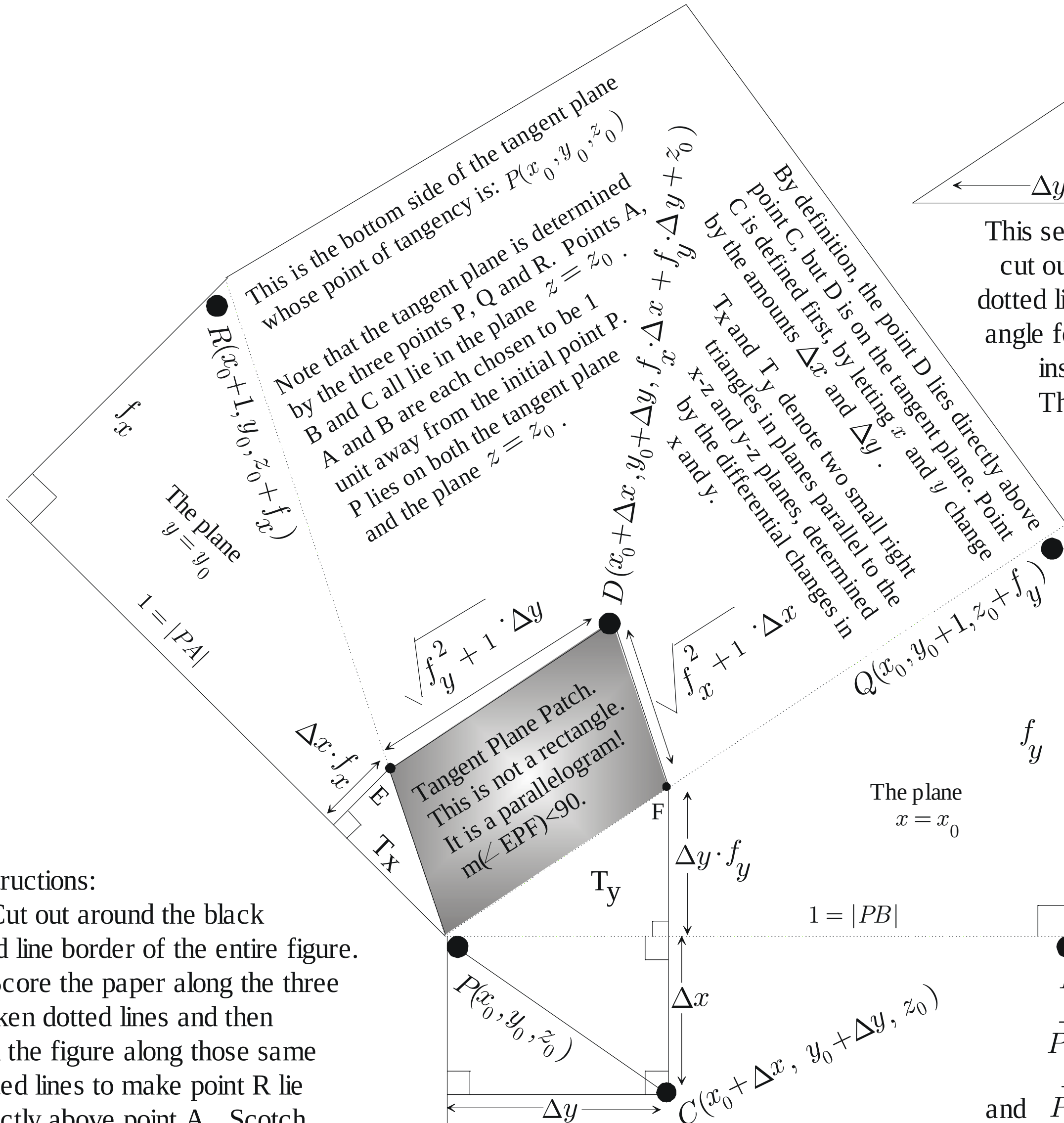


This separate piece can be cut out and folded on the dotted line to make a right-angle fold and then placed inside the larger shape. The vertical line along the right-angle fold connects C to D.



$B(x_0, y_0+1, z_0)$

$\overrightarrow{PQ} = \left\langle 0, 1, f_y \right\rangle$ and $\overrightarrow{PR} = \left\langle 1, 0, f_x \right\rangle$

a normal to the tangent plane

$$= \vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & f_y \\ 1 & 0 & f_x \end{vmatrix} = \left\langle f_x, f_y, -1 \right\rangle = \text{downward normal}$$

$$-\vec{n} = \text{the upper normal} = \left\langle -f_x, -f_y, 1 \right\rangle$$

$$CD = f_x \cdot \Delta x + f_y \cdot \Delta y = dz \approx \Delta z$$

The surface is given by:

$$z = f(x,y)$$
$$z_0 = f(x_0, y_0)$$

$$f_x = f_x(x_0, y_0) \quad f_y = f_y(x_0, y_0)$$

$A(x_0+1, y_0, z_0)$

Therefore, a tangent plane equation is:

$$0 = f_x \cdot (x - x_0) + f_y \cdot (y - y_0) + (-1) \cdot (z - z_0)$$

which simplifies to:

$$(z - z_0) = f_x \cdot (x - x_0) + f_y \cdot (y - y_0)$$

The plane $z = z_0$

Tangent Plane. Differential. Normal Vectors. Surface Patch Area.

Instructions:

1. Cut out around the black solid line border of the entire figure.
2. Score the paper along the three broken dotted lines and then fold the figure along those same dotted lines to make point R lie directly above point A. Scotch tape the two edges which join along the line from point P to point A.
3. The result should be a 3-D tangent plane which lies above the square plane that is shown on the right.
4. Note that the partial derivative heights represented by f_x and f_y are different heights.
5. Note that point C is near point P and the z-coordinate of the tangent plane point D above C can be used to approximate the function value $f(x_0+\Delta x, y_0+\Delta y)$
6. After making both parts, position the separate piece in the back corner so the folded edge connects C and D.

$$CD = f_x \cdot \Delta x + f_y \cdot \Delta y = dz \approx \Delta z$$

The area of the tangent plane parallelogram patch is the length of the vector that is the cross-product of the two vectors:

$$\overrightarrow{PF} = \Delta y \cdot \overrightarrow{PQ} = \left\langle 0,\ \Delta y\ ,\Delta y \cdot f_y \right\rangle$$

$$\overrightarrow{PE} = \Delta x \cdot \overrightarrow{PR} = \left\langle \Delta x\ ,\ 0,\ \Delta x \cdot f_x \right\rangle$$

$$\overrightarrow{PF} \times \overrightarrow{PE} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & \Delta y & \Delta y \cdot f_y \\ \Delta x & 0 & \Delta x \cdot f_x \end{vmatrix}$$

$$= \left\langle f_x \Delta x \Delta y,\ f_y \Delta x \Delta y\ ,-\Delta x \Delta y \right\rangle$$

$$\left\| \overrightarrow{PF} \times \overrightarrow{PE} \right\| = \sqrt{f_x^{\,2} + f_y^{\,2} + 1}\ \ \Delta x\, \Delta y$$

The cosine of the angle between the vectors $\overrightarrow{PE}$ and $\overrightarrow{PF}$ is <1 in absolute value:

$$\frac{\overrightarrow{PE} \cdot \overrightarrow{PF}}{\left\|\overrightarrow{PE}\right\| \cdot \left\|\overrightarrow{PF}\right\|} = \frac{\Delta x f_x \Delta y f_y}{\sqrt{f_x^{\,2}+1}\ \cdot \Delta x \cdot \sqrt{f_y^{\,2}+1}\ \cdot \Delta y} = \frac{f_x \cdot f_y}{\sqrt{f_x^{\,2}+1}\ \cdot \sqrt{f_y^{\,2}+1}}$$