Trochoidal Curves

Includes Hypotrochoids, Epitrochoids, Peritrochoids and Trochoidal Roses

Special cases include Deltoids, Astroids, Cardioids, Nephroids, and Limacons

by

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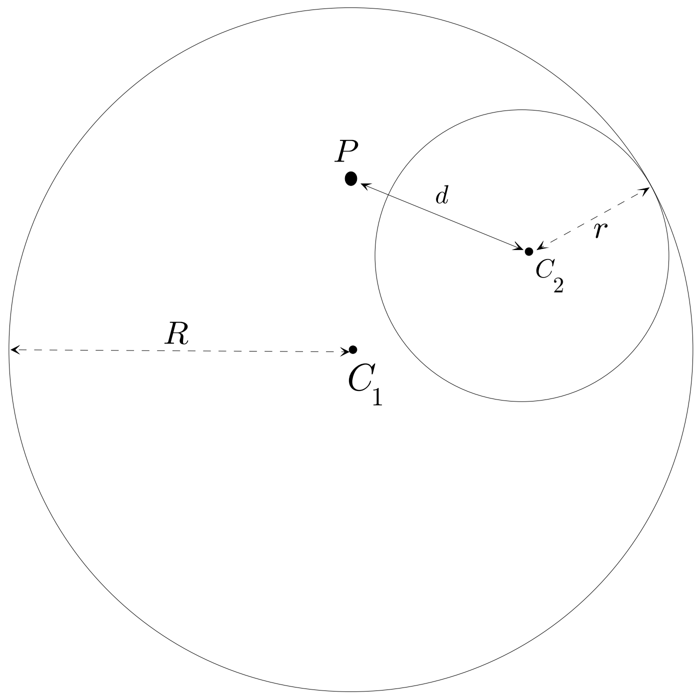
A **Hypotrochoid** is the name of the set of points traced out by a single point P that is considered to be attached by a line connecting it to the center of a small circle of radius r. That smaller circle is then rotated around the inside of a larger circle of radius R, all the time maintaining a point of contact without slipping, so that the smaller circle and larger circle always share a common tangent at their one point of intersection. Let d denote the constant distance between point P and the center of the smaller circle. Then when $d > r$ point P is outside the smaller circle as shown in the next figure below. When $d = r$ the point P is a point on the circumference of the smaller circle. When $d < r$ then point P is a point in the interior of the smaller circle.



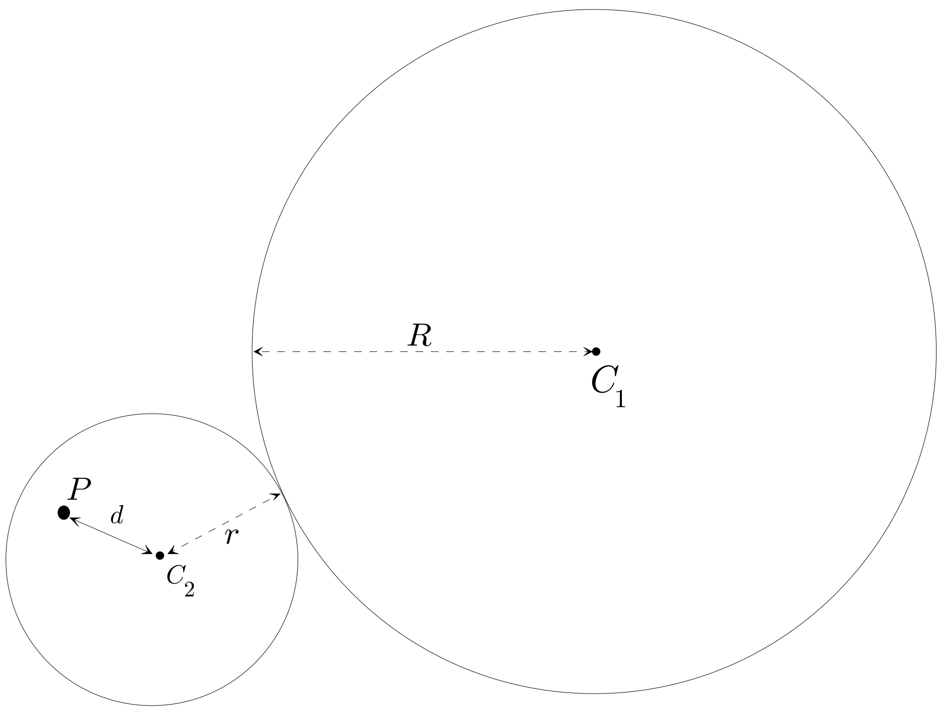
$$x = (R-r)\cdot cos(t) + d\cdot cos\left(\frac{R-r}{r}t\right)$$

$$y = (R-r)\cdot sin(t) - d\cdot sin\left(\frac{R-r}{r}t\right)$$

When $d < r$ the curve is called a curtate hypotrochoid.
When $d = r$ the curve is called a hypocycloid.
When $d > r$ the curve is called a prolate hypotrochoid.

When $d = r$ and and $\frac{R}{r} = 3$ the hypocyloid curve is called a ***deltoid***. When $d = r$ and $\frac{R}{r} = 4$ the hypocycloid curve is called an ***astroid***.

An **Epitrochoid** is the name of the set of points traced out by a single point P that is considered to be attached by a line connecting it to the center of a small circle of radius r. That smaller circle is then rotated around the outside of a larger circle of radius R, all the time maintaining a point of contact without slipping so that the smaller circle and larger circle always share a common tangent at their one point of intersection. Let d denote the constant distance between point P and the center of the smaller circle. Then when $d > r$ point P is outside the smaller circle. When $d = r$ the point P is a point on the circumference of the smaller circle. When $d < r$ then point P is a point in the interior of the smaller circle as is shown in the figure below.



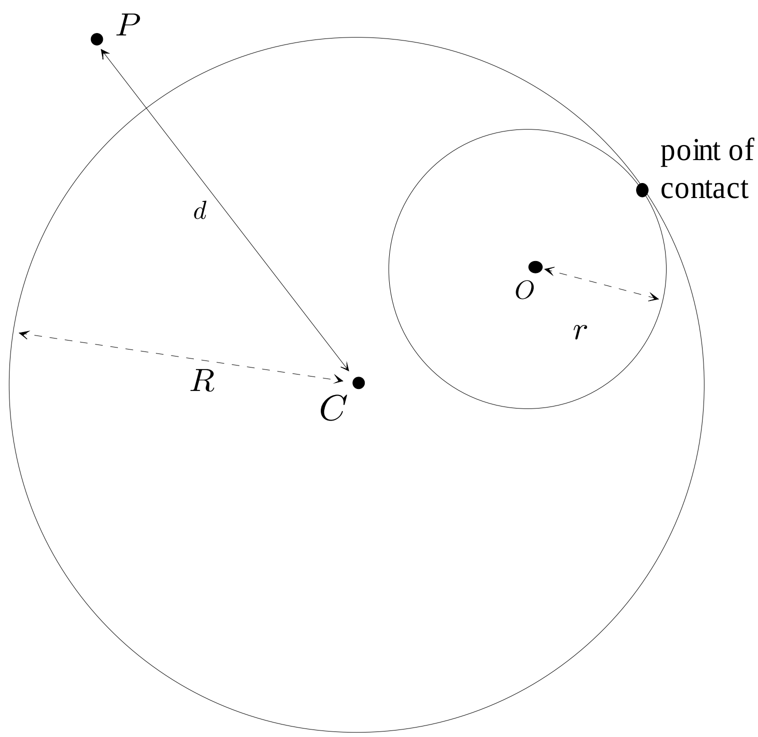
$$x = (R + r) \cdot cos(t) - d \cdot cos\left(\frac{R + r}{r} t\right)$$

$$y = (R + r) \cdot sin(t) - d \cdot sin\left(\frac{R + r}{r} t\right)$$

When $d < r$ the curve is called a curtate epitrochoid.
When $d = r$ the curve is called an epicycloid.
When $d > r$ the curve is called a prolate epitrochoid.

When $d = r$ and $\frac{R}{r} = 1$, the epicycloid curve is called a ***cardioid***. When $d = r$ and $\frac{R}{r} = 2$ the epicycloid curve is called a ***nephroid***.

A **Peritrochoid** is the name of the set of points traced out by a single point P that is considered to be attached to the center point of a large circle of radius R. The large circle is then rotated on its interior edge around a smaller circle of radius r. While the large circle is being rotated, it maintains a point of contact with the small circle, without slipping, and the two circles share a common tangent line at their continual point of contact. Assume the small circle has center O, and assume the large circle has center C and assume the point P is always a distance d from C, even as the large circle moves around the smaller one.

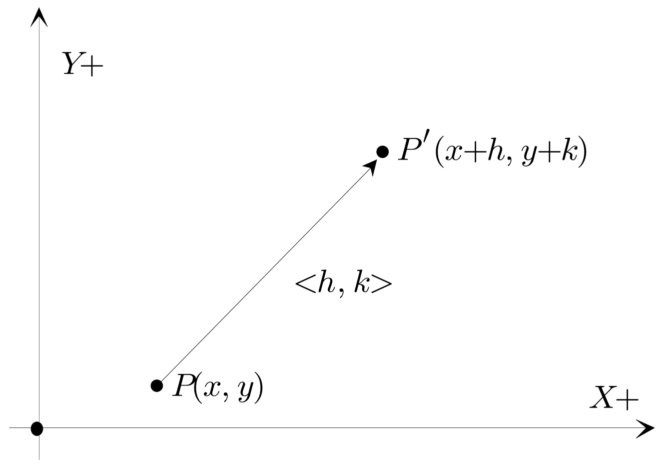


$$x = (r - R) \cdot cos(t) + d \cdot cos\left(\frac{R-r}{R}t\right)$$

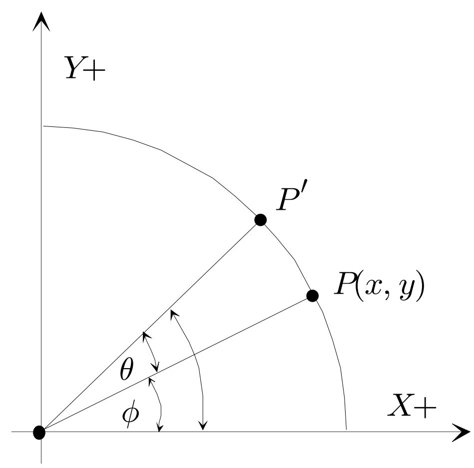
$$y = (r - R) \cdot sin(t) + d \cdot sin\left(\frac{R-r}{R}t\right)$$

When $R = 2r$ the curve is a ***limacon***.

To establish the formulas for the various kinds of figures we need to first understand the effects of rotations and translations on points in the plane. Translations are easiest and may be described by considering the movement of an original point $P(x,y)$ which gets translated by applying the effect of the position vector $< h,k >$. The new point may be called P' and we have $P'(x+h,y+k)$.



Next, consider the rotation of a point under the influence of a counterclockwise angle. In this case, we consider the xy-axes to remain stationary but we consider the points in the plane to rotate.



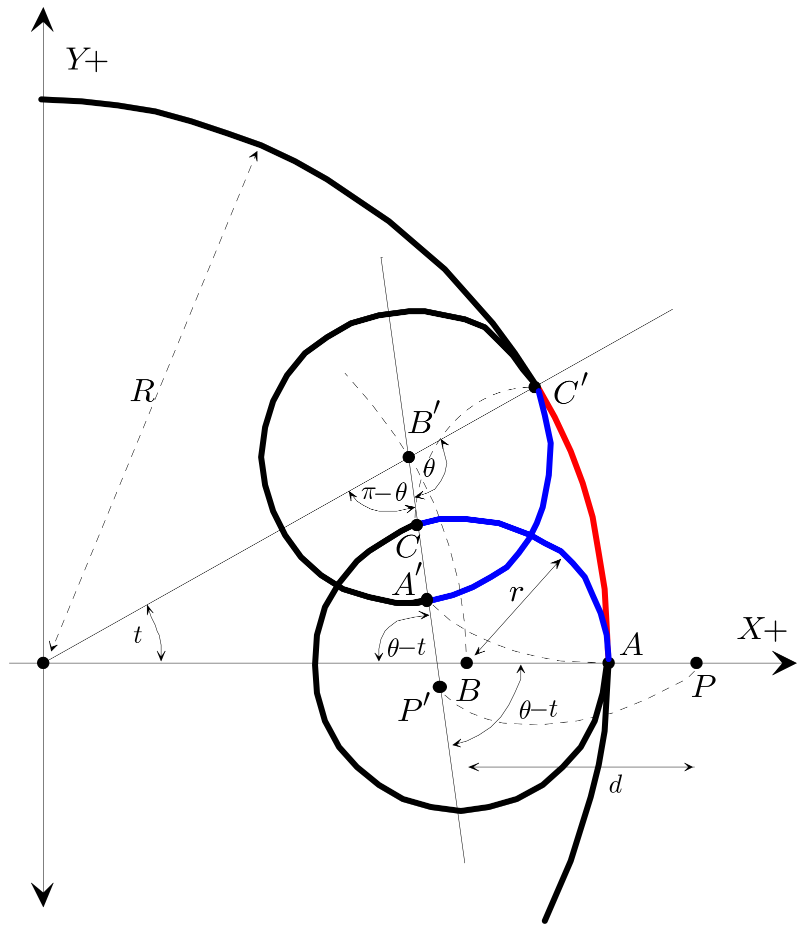
The original point P has coordinates $P(x,y)$. If we let $r = \sqrt{x^2+y^2}$ then we have $x = r \cdot cos(\phi)$ and $y = r \cdot sin(\phi)$. After rotating all points in the plane in the counterclockwise direction with a positive angle θ then the new coordinates of the point P which is now located and labeled as P', is such that

$x' = r \cdot cos(\phi+\theta) = r \cdot [cos(\phi) \cdot cos(\theta) - sin(\phi) \cdot sin(\theta)] =$
$r \cdot cos(\phi) \cdot cos(\theta) - r \cdot sin(\phi) \cdot sin(\theta) =$
$x \cdot cos(\theta) - y \cdot sin(\theta)$
while
$y' = r \cdot sin(\phi+\theta) = r \cdot [\, sin(\phi) \cdot cos(\theta) + cos(\phi) \cdot sin(\theta)] =$
$r \cdot sin(\phi) \cdot cos(\theta) + r \cdot cos(\phi) \cdot sin(\theta) =$
$y \cdot cos(\theta) + x \cdot sin(\theta)$

If the angle of rotation is clockwise, we would just replace θ with $-\theta$.

Hypotrochoid Parametric Formulas Derivation

To establish the parametric formulas for the Hypotrochoid, consider the following figure.



The small circle with center $B(R-r,0)$ is considered to be initially positioned as shown above. Consider the line segment PB as if it is attached to this same small circle. Then we rotate this small circle around the inside of the larger circle of radius R, all the time maintaining a point of contact. Assume the second small circle with center B' represents the new rotated position of the original small circle. Note the coordinates for $B'((R-r)\cdot cos(t),\ (R-r)\cdot sin(t))$. In fact both B and B' are points on another circle of radius $R-r$ and center at the origin. We are primarily interested in the new coordinates of point P when it changes from its initial position, $P(R-r+d,0)$, to the new point P'. In the above figure we show the points A and C as points on the original small circle, and we show the paths these points take during the rotation, and we show their final dispositions as the points A' and C'. We also show the path point P takes when it moves to its new position at P'. In the above figure, the blue arc AC moves and becomes the new blue arc $A'C'$. These two blue arcs have the same arclength as the red arc AC', but the red arc AC' is on the large circle of radius R. Equating these arc lengths means that $Rt = r\theta$ so that $\theta = \dfrac{R}{r}t$. Since $R > r$, this last equation implies $\theta > t$. In the above figure the ratio of $\dfrac{R}{r}$ is about $4:1$. This means the angle θ is about 4 times as large as angle t.

Now we can describe the transformations any point like P goes through when that point is considered attached to the small circle and the point gets transformed by the inside rolling action of the small circle.

1. Start with the original coordinates of any point attached to the small circle.

2. Translate the point using the vector $< -(R-r), 0 >$ which has the effect of moving point B to the origin.

3. Rotate the new point by a clockwise angle $\theta - t = \frac{R}{r}t - t = t\left(\frac{R}{r} - 1\right) = \frac{R-r}{r}t.$

4. Finally translate the point using the vector $< (R-r)\cdot cos(t),\ (R-r)\cdot sin(t) >$ which has the effect of moving the origin point to $B\,'$.

Let's apply the above transformations to the original P point to arrive at the new coordinates of $P\,'$.

1. $P((R-r)+d, 0)$.

2. $(d, 0)$.

3. $(d\cdot cos(\theta - t) + 0\cdot sin(\theta - t),\ 0\cdot cos(\theta - t) - (d)\cdot sin(\theta - t)) =$

   $(d\cdot cos(\theta - t), -d\cdot sin(\theta - t))$.

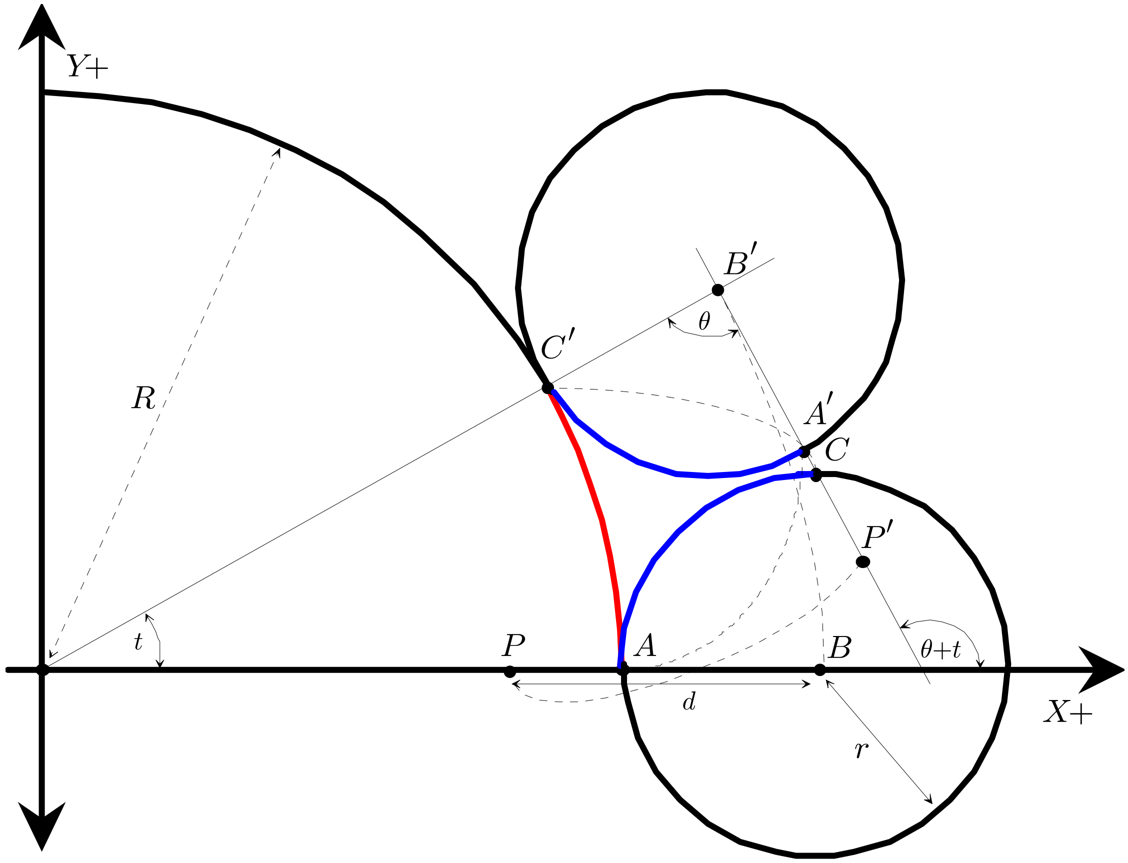
4. $((R-r)\cdot cos(t) + d\cdot cos(\theta - t),\ (R-r)\cdot sin(t) - d\cdot sin(\theta - t)) =$

   $$\left((R-r)\cdot cos(t) + d\cdot cos\left(\frac{R-r}{r}t\right),\ (R-r)\cdot sin(t) - d\cdot sin\left(\frac{R-r}{r}t\right)\right) =$$

That's all there is to understanding the parametric equations for the Hypotrochoid.

Epitrochoid Parametric Formulas Derivation

In the figure below, we assume we have a large circle of radius R that is centered at the origin. A smaller circle of radius r is given a starting position with its center at point B on the x-axis. This smaller circle shares a common tangent with the large circle at point A. Then the smaller circle is rotated outside the large circle, all the while maintaining a point of contact with the larger circle. The points A, B, C shown below get transformed to the new points $A', B\,', C'$ by moving along the paths indicated when the smaller circle is rotated in a counterclockwise direction. We wish to study the position and path that the point P takes when the small circle is rotated around the outside of the larger circle. Point P gets transformed and becomes the new point $P\,'$.



We assume the arclengths of the two blue arcs, namely arc AC and the arc $A'C'$ are the same length and both are equal in length to the red arc AC'. Of course the two blue arcs are on the small circle, while the red arc is on the large circle. Since these arclengths are the same we know $Rt = r\theta$ so that $\theta = \frac{R}{r}t$. This means the fraction $\frac{R}{r}$ acts as a multiplier on the angle t.

Now we can describe the transformations any point like P goes through when that point is considered attached to the small circle and the point gets transformed by the outside rolling action of the small circle.

1. Start with the original coordinates of any point attached to the small circle.
2. Translate the point using the vector $< -(R+r), 0 >$ which has the effect of moving point B to the origin.
3. Rotate the new point by a clockwise angle $\theta + t = \dfrac{R}{r}t + t = t\left(\dfrac{R}{r} + 1\right) = \dfrac{R+r}{r}t$.
4. Finally translate the point using the vector $< (R+r) \cdot cos(t), \, (R+r) \cdot sin(t) >$ which has the effect of moving the origin point to $B\,'$.

Let's apply the above transformations to the original P point to arrive at the new coordinates of P'.

1. $P((R+r)-d,0)$.

2. $(-d,0)$.

3. $(-d \cdot cos(\theta+t) - 0 \cdot sin(\theta+t),\ 0 \cdot cos(\theta+t) + (-d) \cdot sin(\theta+t)) =$

$(-d \cdot cos(\theta+t), -d \cdot sin(\theta+t))$.

4. $((R+r) \cdot cos(t) - d \cdot cos(\theta+t),\ (R+r) \cdot sin(t) - d \cdot sin(\theta+t)) =$

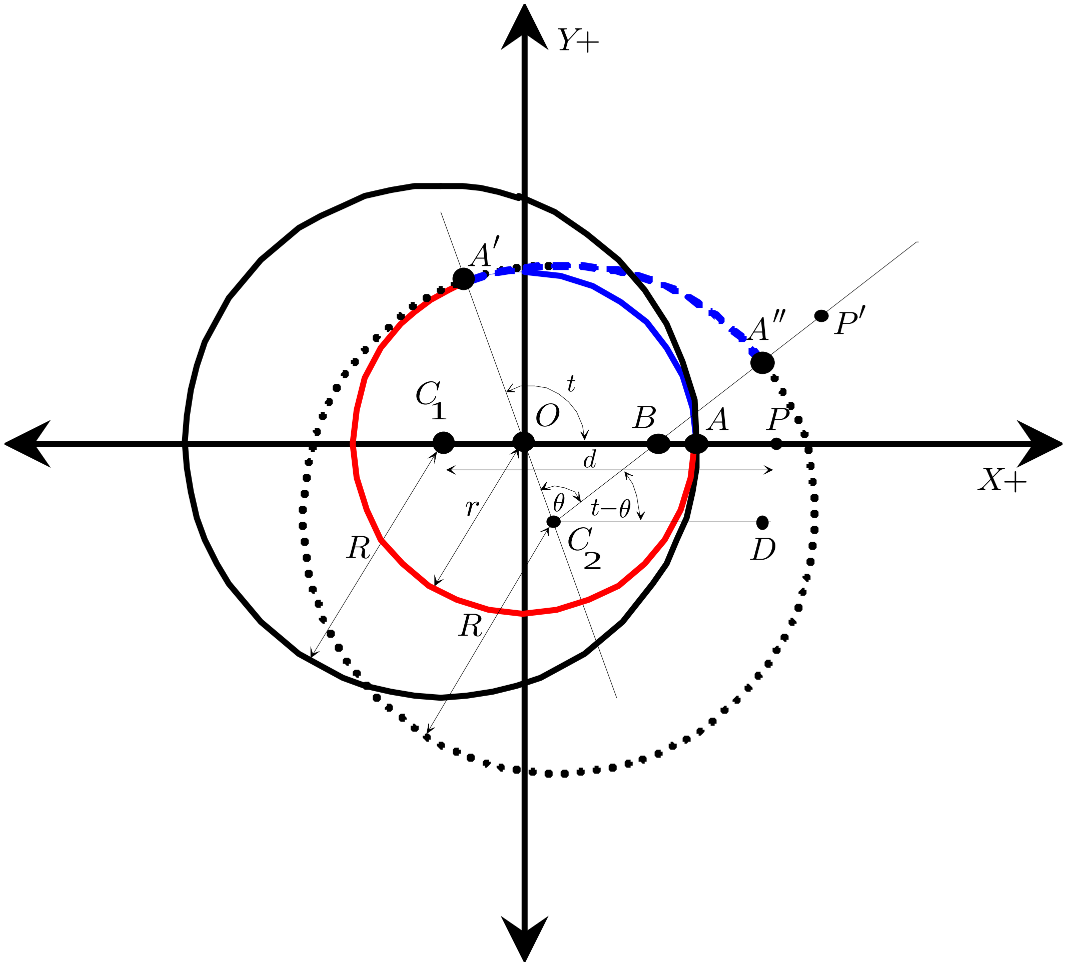
$$\left((R+r) \cdot cos(t) - d \cdot cos\left(\frac{R+r}{r}t\right),\ (R+r) \cdot sin(t) - d \cdot sin\left(\frac{R+r}{r}t\right)\right)$$

That's all there is to understanding the parametric equations for the Epitrochoid.

Peritrochoid Parametric Formulas Derivation

To understand the parametric equations for the Peritrochoid consider the next figure in which we have a small circle of radius r, centered at the origin point O of an xy-coordinate system. The points A and A' are on that small circle. A large circle of radius R is positioned such that its center is at C_1 and it shares point A as an initial common point of contact. At point A we could consider the two circles to share a common vertical tangent line.

Now let P denote the point that initially starts on the x-axis, where P is a distance d from the center C_1 of the larger circle. In other words, P is d units to the right of C_1. Now the small circle of radius r that is centered at the origin remains fixed, while it is the larger circle that will be moved and rotated around the small circle, without slipping. As the large circle is turned in a counterclockwise direction, its center point moves, which causes P to move, but the large circle always maintains exactly one point of contact with the small circle. In the figure below we show a small amount of rotation where the point of contact initially starts at A and rotates on the blue arc of the small circle until point A reaches the point A'. The small circle is mostly red, except for the blue contact arc.



The measure of the central angle in the small circle is denoted by t in the above figure, i.e., the measure of $\angle BOA'$ is t. Because of the common point of contact, the blue arc AA' on the small circle has the same arclength as the blue arc shown in the above figure that extends along the broken line of the final large circle of radius R. That final arc is the one that extends from point A'' to point A'. The final large circle has the point C_2 as its center. Relating the two blue arcs we have $rt = R\theta$, or $\theta = \dfrac{r}{R}t$. Since $R > r$ we will also have $\theta < t$. The fraction $\frac{r}{R}$ acts as a multiplier on the angle t to produce the angle θ.

In the above figure the point C_2 is the center of the final position of the large circle of radius R. Note that the line C_2A'' makes an angle of $t - \theta$ with the horizontal line C_2D extending to the right of C_2. That is because that horizontal line is parallel to the x-axis so the measure of $\angle OC_2D$ is the same as the measure of $\angle A'OB$ which is t. But we know $\angle OC_2B$ measures θ, and this forces $\angle BC_2D$ to measure as $t - \theta$. Also note that the measure of $\angle BOC_2$ is $\pi - t$. This same angle, measured clockwise from the positive x-axis would be the positive number $\pi - t$, which in turn means that measured counterclockwise the angle is really $t - \pi$. The distance from O to C_2 is the same as the distance from O to C_1, namely, $R - r$. Thus the xy-coordinates of point C_2 correspond to the endpoint of the vector $< (R-r)\cdot cos(t-\pi), (R-r)\cdot sin(t-\pi) >$.

We can now describe the transformation of any point in the xy-plane that is considered attached to the center of the large circle by considering how to move the large circle from its starting position, shown in the above figure with a solid circular line, compared with its final resting position shown above with a broken circular line. The following are the required transformations.

1. Start with the original coordinates of any point.

2. Translate the point using the vector $< (R-r), 0 >$ which has the effect of moving the point C_1 to the origin.

3. Rotate the large circle by a counterclockwise angle of measure $t-\theta$.
   $t-\theta = t - \frac{r}{R}t = t\left(1-\frac{r}{R}\right) = \frac{R-r}{R}t.$

4. Finally translate using the vector $< (R-r)\cdot cos((t-\pi)),\ (R-r)\cdot sin((t-\pi)) >$
   $= \ < -(R-r)\cdot cos(t),\ -(R-r)\cdot sin(t) >$
   which has the effect of moving the origin point to the point C_2.

Let's apply the above transformations to the original P point to arrive at the new coordinates of P'.

1. $P(-(R-r)+d, 0).$

2. $(d, 0).$

3. $(d\cdot cos(t-\theta) - 0\cdot sin(t-\theta),\ 0\cdot cos(t-\theta) + d\cdot sin(t-\theta)) =$

   $$\left(d\cdot cos\left(\frac{R-r}{r}\right),\ d\cdot sin\left(\frac{R-r}{r}\right)\right)$$

4. $$\left(-(R-r)\cdot cos(t) + d\cdot cos\left(\frac{R-r}{r}t\right),\ -(R-r)\cdot sin(t) + d\cdot sin\left(\frac{R-r}{r}t\right)\right) =$$

   $$\left((r-R)\cdot cos(t) + d\cdot cos\left(\frac{R-r}{r}t\right),\ (r-R)\cdot sin(t) + d\cdot sin\left(\frac{R-r}{r}t\right)\right)$$

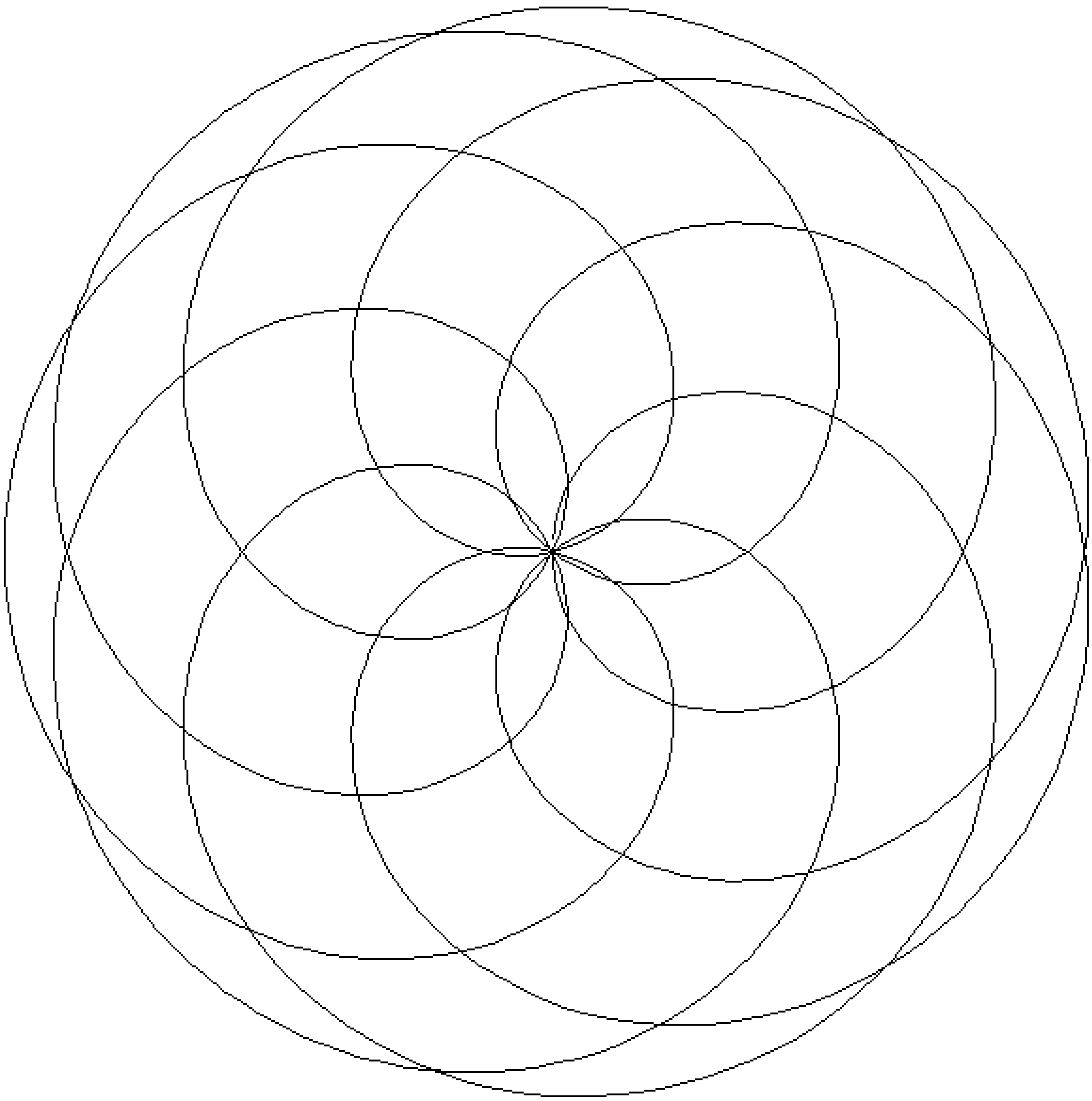
That's all there is to understanding the parametric equations for the Peritrochoid.

A **Trochoidal Rose** is the name of a curve whose parametric equations are:

$x = R(1+r)cos(rt) - R(1+r) \cdot cos(t(1+r))$

$y = R(1+r) \cdot sin(rt) - R(1+r) \cdot sin(t(1+r))$

Note that the above equations only involve the two parameters R and r. The graph below shows the curve when $R = 1$ and $r = \frac{3}{7}$.



The author of this paper has written a free computer program named **Koch** that can be used to graph all of the curves described in this paper. Even more significant is that this program can be used to make artistic figures by varying parameters that are associated with trochoidal curves. Some examples of these artistic figures are shown below and on the next page.

In the **Koch** program, the value of r is multiplied by the MultiGraph Multiplier for each new graph that gets made when the Trochoidal Rose selection is active. Otherwise, for Hypotrochoids, Epitrochoids, and Peritrochoids it is the d value that gets multiplied by the MultiGraph Multiplier for each new graph that is generated using the MultiGraph number of graphs. To download the **Koch** program visit the Web site: **homepage.smc.edu/kennedy_john** and look for the downloadable software link.