Synthetic Substitution Examples and Polynomial Theorems

Consider $p(x) = 2x^4 + 9x^3 + 4x^2 - 8x + 10$.

What is $p(-3)$?

$$\begin{array}{rrrrr|l}
 & & & & & \underline{-3} \\
2 & 9 & 4 & -8 & 10 & \\
 & -6 & -9 & 15 & -21 & \\ \cline{1-5}
2 & 3 & -5 & 7 & -11 &
\end{array}$$

What is $p(2)$?

$$\begin{array}{rrrrr|l}
 & & & & & \underline{2} \\
2 & 9 & 4 & -8 & 10 & \\
 & 4 & 26 & 60 & 104 & \\ \cline{1-5}
2 & 13 & 30 & 52 & 114 &
\end{array}$$

Note that $p(x) = (((2x + 9)x + 4)x - 8)x + 10$.

If we substitute either $x = -3$ or $x = 2$ into the parenthesized expression we can see that the numbers that appear in the 2nd and 3rd rows of the Synthetic Substitution process really come out of these direct substitutions and calculations. It is for this reason that we prefer to use the term Synthetic Substitution as opposed to Synthetic Division whenever we calculate and fill in a table like those above. In fact, it is most instructive to directly substitute x instead of a number to fill in a Synthetic Substitution table like the one shown below. Counting the number of x's that follow each coefficient in parentheses in the last entry shows that the final polynomial is the same as $p(x) = 2x^4 + 9x^3 + 4x^2 - 8x + 10$.

$$\begin{array}{rrrrr|l}
 & & & & & \underline{x} \\
2 & 9 & 4 & -8 & 10 & \\
 & 2x & (2x{+}9)x & ((2x{+}9)x{+}4)x & (((2x{+}9)x{+}4)x{-}8)x & \\ \cline{1-5}
2 & 2x{+}9 & (2x{+}9)x{+}4 & ((2x{+}9)x{+}4)x{-}8 & (((2x{+}9)x{+}4)x{-}8)x{+}10 &
\end{array}$$

Now consider the long division of $p(x)$ by $x + 3$.

$$\begin{array}{rrrrrrrrrrrrr}
 & & & & 2x^3 & + & 3x^2 & - & 5x & + & 7 & & \\ \cline{4-13}
x & + & 3 & \big) & 2x^4 & + & 9x^3 & + & 4x^2 & - & 8x & + & 10 \\
 & & & & 2x^4 & + & 6x^3 & & & & & & \\ \cline{4-7}
 & & & & & & 3x^3 & + & 4x^2 & & & & \\
 & & & & & & 3x^3 & + & 9x^2 & & & & \\ \cline{7-9}
 & & & & & & & & -5x^2 & - & 8x & & \\
 & & & & & & & & -5x^2 & - & 15x & & \\ \cline{9-11}
 & & & & & & & & & & 7x & + & 10 \\
 & & & & & & & & & & 7x & + & 21 \\ \cline{11-13}
 & & & & & & & & & & & & -11
\end{array}$$

Note the coefficients in the quotient polynomial are the same as the first numbers in the last row of the first table above. Also note that the remainder is -11 and this is the last number in the third row of the first table above. These are reasons why some people use the term Synthetic Division.

Next consider the polynomial that is the product of the following 4 linear factors:

$$p(x) = (x+2)(x-3)(4x+1)(3x-5) = (x^2-x-6)(12x^2-17x-5)$$

$$= 12x^4 - 29x^3 - 60x^2 + 107x + 30$$

$$\begin{array}{rrrrr|l} & & & & & \underline{-2} \\ 12 & -29 & -60 & 107 & 30 & \\ & -24 & 106 & -92 & -30 & \\ \hline 12 & -53 & 46 & 15 & 0 & \end{array}$$

$$\begin{array}{rrrrr|l} & & & & & \underline{3} \\ 12 & -29 & -60 & 107 & 30 & \\ & 36 & 21 & -117 & -30 & \\ \hline 12 & 7 & -39 & -10 & 0 & \end{array}$$

$$\begin{array}{rrrrr|l} & & & & & \underline{-1/4} \\ 12 & -29 & -60 & 107 & 30 & \\ & -3 & 8 & 13 & -30 & \\ \hline 12 & -32 & -52 & 120 & 0 & \end{array}$$

$$\begin{array}{rrrrr|l} & & & & & \underline{5/3} \\ 12 & -29 & -60 & 107 & 30 & \\ & 20 & -15 & -125 & -30 & \\ \hline 12 & -9 & -75 & -18 & 0 & \end{array}$$

Note that the product of the four roots is: $-2 \cdot 3 \cdot \left(-\frac{1}{4}\right) \cdot \left(\frac{5}{3}\right) = \frac{+30}{12}$.

The sum of the four roots is: $-2 + 3 + \frac{-1}{4} + \frac{5}{3} = \frac{12}{12} + \frac{-3}{12} + \frac{20}{12} = \frac{29}{12} = -\left(\frac{-29}{12}\right)$.

The next computations illustrate the analysis of ***Upper and Lower bounds***.

$$\begin{array}{rrrrr|l} & & & & & \underline{-3} \\ 12 & -29 & -60 & 107 & 30 & \\ & -36 & 195 & -405 & 894 & \\ \hline 12 & -65 & 135 & -298 & 924 & \end{array}$$

$$\begin{array}{rrrrr|l} & & & & & \underline{-4} \\ 12 & -29 & -60 & 107 & 30 & \\ & -48 & 308 & -992 & 3540 & \\ \hline 12 & -77 & 248 & -885 & 3570 & \end{array}$$

$$\begin{array}{rrrrr|l} & & & & & \underline{4} \\ 12 & -29 & -60 & 107 & 30 & \\ & 48 & 76 & 64 & 684 & \\ \hline 12 & 19 & 16 & 171 & 714 & \end{array}$$

$$\begin{array}{rrrrr|l} & & & & & \underline{5} \\ 12 & -29 & -60 & 107 & 30 & \\ & 60 & 155 & 475 & 2910 & \\ \hline 12 & 31 & 95 & 582 & 2940 & \end{array}$$

Next, consider the polynomial $p(x) = (x+5)(x^2-4x-2) = x^3+x^2-22x-10$

$$\begin{array}{rrrrl} & & & & \lfloor\underline{-5} \\ 1 & 1 & -22 & -10 & \\ & -5 & 20 & 10 & \\ \hline 1 & -4 & -2 & 0 & \end{array}$$

We can also substitute with ***irrational numbers***.

$$\begin{array}{rrrrl} & & & & \lfloor\underline{2-\sqrt{6}} \\ 1 & 1 & -22 & -10 & \\ & 2-\sqrt{6} & 12-5\sqrt{6} & 10 & \\ \hline 1 & 3-\sqrt{6} & -10-5\sqrt{6} & 0 & \end{array} \qquad \begin{array}{rrrrl} & & & & \lfloor\underline{2+\sqrt{6}} \\ 1 & 1 & -22 & -10 & \\ & 2+\sqrt{6} & 12+5\sqrt{6} & 10 & \\ \hline 1 & 3+\sqrt{6} & -10+5\sqrt{6} & 0 & \end{array}$$

Note that the product of the three roots is:

$$(-5)\left(2-\sqrt{6}\right)\left(2+\sqrt{6}\right) = (-5)(4-6) = (-5)(-2) = 10 = (-1)^3 \cdot (-10).$$

The sum of the roots is $-5+4 = -1 = -(+1)$.

Next, consider the polynomial $p(x) = (x-2)(x^2-6x+25) = x^3-8x^2+37x-50$

$$\begin{array}{rrrrl} & & & & \lfloor\underline{2} \\ 1 & -8 & 37 & -50 & \\ & 2 & -12 & 50 & \\ \hline 1 & -6 & 25 & 0 & \end{array}$$

We can also substitute with ***complex numbers***.

$$\begin{array}{rrrrl} & & & & \lfloor\underline{3+4i} \\ 1 & -8 & 37 & -50 & \\ & 3+4i & -31-8i & 50 & \\ \hline 1 & -5+4i & 6-8i & 0 & \end{array} \qquad \begin{array}{rrrrl} & & & & \lfloor\underline{3-4i} \\ 1 & -8 & 37 & -50 & \\ & 3-4i & -31+8i & 50 & \\ \hline 1 & -5-4i & 6+8i & 0 & \end{array}$$

Note that the product of the three roots is:

$$2(3-4i)(3+4i) = 2[9+16] = 2[25] = 50 = (-1)^3(-50)$$

The sum of the three roots is: $2+3+3 = 8 = -(-8)$.

The following 31 items summarize some of the most important theorems about polynomials. See the note at the end of this list if you are interested in reading any of the detailed proofs of these theorems.

1. The Division Algorithm

If $p(x)$ and $d(x) \not\equiv 0$ are any two polynomials then there exist unique polynomials $q(x)$ and $r(x)$ such that $p(x) = d(x) \cdot q(x) + r(x)$ where the degree of $r(x)$ is stricly less than the degree of $d(x)$ when the degree of $d(x) \geq 1,$ or else $r(x) \equiv 0.$

2. The Division Check for a Linear Divisor

Consider dividing the polynomial $p(x)$ by the linear term $(x - a).$ Then, the ***Division Check*** states that: $p(x) = (x - a) \cdot q(x) + r$

3. Remainder Theorem

When any polynomial $p(x)$ is divided by $(x - a)$ the remainder is $p(a).$

4. Factor Theorem

$(x - a)$ is a factor of the polynomial $p(x)$ if and only if $p(a) = 0.$

5. Maximum Number of Zeros Theorem

A polynomial cannot have more real zeros than its degree.

6. Fundamental Theorem of Algebra

a) Every polynomial of degree $n \geq 1$ has at least one zero among the complex numbers.

b) If $p(x)$ denotes a polynomial of degree $n,$ then $p(x)$ has exactly n roots, some of which may be either irrational numbers or complex numbers.

7. Product and Sum of the Roots Theorem

Let $p(x) = 1x^n + a_{n-1}x^{n-1} + \cdots + a_3x^3 + a_2x + a_1x + a_0$ be any polynomial with real coefficients with a ***leading coefficient of 1*** where $n \geq 1.$ Then a_0 is $(-1)^n$ times the product of all the roots of $p(x) = 0$ and a_{n-1} is the opposite of the sum of all the roots of $p(x) = 0.$

8. Rational Roots Theorem

Let $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_3x^3 + a_2x^2 + a_1x + a_0$ be any polynomial with integer coefficients. If the reduced rational number $\dfrac{c}{d}$ is a root of $p(x) = 0$ then c must be a factor of a_0 and d must be a factor of $a_n.$

9. Integer Roots Theorem

Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_3x^3 + a_2x^2 + a_1x + a_0$ be any polynomial with integer coefficients and ***with a leading coefficient of 1***. If $p(x)$ has any rational zeros, then those zeros must all be integers.

10. Upper and Lower Bounds Theorem

Let $p(x)$ be any polynomial with ***real coefficients*** and ***a positive leading coefficient*.**

(***Upper Bound***) If $a > 0$ and $p(a) > 0$ and if in applying synthetic substitution to compute $p(a)$ all numbers in the 3rd row are positive, then a is an upper bound for all the roots of $p(x) = 0.$

(***Lower Bound***) If $a < 0$ and $p(a) \neq 0$ and if in applying synthetic substitution to compute $p(a)$ all the numbers in the 3rd row alternate in sign then a is a lower bound for all the roots of $p(x) = 0.$

[In either bound case, we can allow any number of zeros in any positions in the third row except in the first and last positions. The first number is assumed to be positive and the last number is $p(a) \neq 0$. For upper bounds, we can state alternatively and more precisely that no negatives are allowed in the 3rd row. In the lower bound case the alternating sign requirement is not strict either, as any 0 value can assume either sign as required. In practice you may rarely see any zeros in the 3rd row. However, a slightly stronger and more precise statement is that the bounds still hold even when zeros are present anywhere as interior entries in the 3rd row.]

11. Intermediate Value Theorem

If $p(x)$ is any polynomial with ***real coefficients***, and if $p(a) > 0$ and $p(b) < 0$ then there is at least one real number c between a and b such that $p(c) = 0.$

12. Single Bound Theorem

Let $p(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_3x^3 + a_2x^2 + a_1x + a_0$ be any polynomial with real coefficients and a leading coefficient of 1. Let $M_1 = 1 + max\{|a_0|, |a_1|, |a_2|, \ldots, |a_{n-1}|\}$ and let $M_2 = max\{1, |a_0| + |a_1| + |a_2| + \ldots + |a_{n-1}|\}$. Finally let $M = min\{M_1, M_2\}$. Then every zero of $p(x)$ lies between $-M$ and M.

13. Odd Degree Real Root Theorem

If $p(x)$ has real coefficients and has a degree that is odd then it has at least one real root.

14. Complex Conjugate Roots Theorem

If $p(x)$ is any polynomial with ***real coefficients***, and if $a + bi$ is a complex root of the equation $p(x) = 0$, then another complex root is its conjugate $a - bi.$

(Complex number roots appear in conjugate pairs)

15. Linear and Irreducible Quadratic Factors Theorem

Let $p(x)$ be any polynomial with real coefficients. Then $p(x)$ may be written as a product of linear factors and irreducible quadratic factors. The sum of all the degrees of these component factors is the degree of $p(x).$

16. Irrational Conjugate Roots Theorem

Let $p(x)$ be any polynomial with ***rational*** real coefficients. If $a + b\sqrt{c}$ is a root of the equation $p(x) = 0$ where $\sqrt{c}$ is irrational and a and b are rational, then another root is $a - b\sqrt{c}$.

(Like complex roots, irrational real roots appear in conjugate pairs, but only when the polynomial has rational coefficients.)

17. Descartes's Rule of Signs Lemma 1.

If $p(x)$ has real coefficients, and if $p(a) = 0$ where $a > 0$, then $p(x)$ has at least one more sign variation than the quotient polynomial $q(x)$ has sign variations where $p(x) = (x - a)q(x)$. [When the difference in the number of sign variations is greater than 1, the difference is always an odd number.]

18. Descartes's Rule of Signs Lemma 2.

If $p(x)$ has real coefficients, the number of positive zeros of $p(x)$ is not greater than the number of variations in sign of the coefficients of $p(x)$.

19. Descartes's Rule of Signs Lemma 3.

Let $r_1, r_2, r_3, \ldots, r_k$ denote k positive numbers and let $p(x) = \prod_{i=1}^{k}(x - r_i)$.

Then the coefficients of $p(x)$ are all alternating in sign and this polynomial has exactly k sign variations in its coefficients.

20. Descartes's Rule of Signs Lemma 4.

The number of variations in sign of a polynomial with real coefficients is even if the first and last coefficients have the same sign, and is odd if the first and last coefficients have opposite signs.

21. Descartes's Rule of Signs Lemma 5.

If the number of positive zeros of $p(x)$ with real coefficients is less than the number of sign variations in $p(x)$, it is less by an even number.

22. Descartes's Rule of Signs Lemma 6.

Each negative root of $p(x)$ corresponds to a positive root of $p(-x)$. That is, if $a < 0$ and a is a zero of $p(x)$, then $-a$ is a positive zero of $p(-x)$.

23. Descartes's Rule of Signs

Let $p(x)$ be any polynomial with ***real coefficients***.

(***Positive Roots***) The number of positive roots of $p(x) = 0$ is either equal to the number of sign variations in the coefficients of $p(x)$ or else is less than this number by an even integer.

(***Negative Roots***) The number of negative roots of $p(x) = 0$ is either equal to the number of sign variations in the coefficients of $p(-x)$ or else is less than this number by an even integer.

Note that when determining sign variations we can ignore terms with zero coefficients.

24. Lemma On Continuous Functions.

Let $f(x)$ and $g(x)$ be two continuous real-valued functions with a common domain that is an open interval $(a,\, b)$. Furthermore let $c \in (a,\, b)$ and assume that except when $x = c$ we have $f(x) = g(x)$ for all $x \in (a,\, b)$. Then we must also have $f(c) = g(c)$.

25. Theorem On the Equality of Polynomials

Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_3 x^3 + a_2 x^2 + a_1 x + a_0$ and let

$q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_3 x^3 + b_2 x^2 + b_1 x + b_0$ be any two real polynomials of

degrees n and m respectively. If for all real numbers x, $p(x) = q(x)$ then

1) $m = n$

and 2) for all i, if $0 \leq i \leq n$ then $a_i = b_i$.

26. Theorem Euclidean Algorithm for Polynomials

Let $p(x)$ and $q(x)$ be any two polynomials with degrees ≥ 1. Then there exists a polynomial $d(x)$ such that $d(x)$ divides evenly into both $p(x)$ and $q(x)$. Moreover, $d(x)$ is such that if $a(x)$ is any other common divisor of $p(x)$ and $q(x)$, then $a(x)$ divides evenly into $d(x)$. The polynomial $d(x)$ is called the Greatest Common Divisor of $p(x)$ and $q(x)$ is sometimes denoted by $GCD(p(x), q(x))$. Except for constant multiples, $d(x)$ is unique.

27. Corollary to the Euclidean Algorithm for Polynomials

The GCD of any two polynomials $p(x)$ and $q(x)$ may be expressed as a linear combination of $p(x)$ and $q(x)$.

28. Lemma 1 for Partial Fractions

If $f(x) = \dfrac{a(x)}{b(x)c(x)}$ where $GCD(b(x), c(x)) = 1$ then there exist polynomials $d(x)$ and $e(x)$ such that

$$f(x) = \frac{d(x)}{b(x)} + \frac{e(x)}{c(x)}$$

29. Lemma 2 for Partial Fractions

If $f(x) = \dfrac{p(x)}{[\,q(x)\,]^m}$ then there exists a polynomial $g(x)$ and for $1 \leq i \leq m$ there exist polynomials $s_i(x)$ each with degree less than $q(x)$ such that

$$f(x) = \frac{p(x)}{[\,q(x)\,]^m} = g(x) + \frac{s_1(x)}{q(x)} + \frac{s_2(x)}{[\,q(x)\,]^2} + \frac{s_3(x)}{[\,q(x)\,]^3} + \cdots + \frac{s_m(x)}{[\,q(x)\,]^m}$$

30. Partial Fraction Decomposition Theorem

Let $\dfrac{p(x)}{q(x)}$ be a rational function where $p(x)$ and $q(x)$ are polynomials such that the degree of $p(x)$ is less than the degree of $q(x)$. Then there exist algebraic fractions $F_1, F_2, \ldots, F_r$ such that $\dfrac{p(x)}{q(x)} = F_1 + F_2 + \cdots + F_r$ and where each F_i fraction is one of two forms: $\dfrac{A_i}{(a_i x + b_i)^{n_i}}$ or $\dfrac{A_k x + B_k}{(a_k x^2 + b_k x + c_k)^{m_k}}$ where $A_i,\ a_i,\ b_i,\ A_k,\ B_k,\ a_k, b_k, c_k$ are all real numbers and the n_i and the m_k are positive integers and each quadratic expression $a_k x^2 + b_k x + c_k$ has a negative discriminant.

31. Partial Fraction Decomposition Coefficient Theorem

Let $\dfrac{p(x)}{q(x)}$ be a rational function where $p(x)$ and $q(x)$ are polynomials such that the degree of $p(x)$ is less than the degree of $q(x)$. If $x = a$ is a root of $q(x) = 0$ of multiplicity 1, then in the partial fraction decomposition of $\dfrac{p(x)}{q(x)}$ which contains a term of the form $\dfrac{A}{(x-a)}$, the constant $A = \dfrac{p(a)}{q'(a)}$.

Detailed proofs of all the above theorems may be found on the author's web site:

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in a 31-page paper titled Some Polynomial Theorems. See the section on the author's web site that is titled Downloadable Papers.