Some Very Special
Trigonometric Function Values

by

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Some Very Special Trigonometric Function Values

This paper will develop the following special trigonometric function values. In addition, we will show how some of these values relate to the famous number known as the Golden Ratio. In turn, we will make connections to both the area and angles of a pentagon and we will further see the importance of these values and how they relate to 3-dimensional regular polyhedra such as the dodecahedron and the icosahedron. These special angles are simply all the multiples of 9° that do not include the easy angles of $0^{\circ}, 45^{\circ}$, and 90°.

$$sin(9^{\circ}) = \frac{\sqrt{8-2\sqrt{10+2\sqrt{5}}}}{4} = cos(81^{\circ})$$

$$sin(18^{\circ}) = \frac{-1+\sqrt{5}}{4} = cos(72^{\circ})$$

$$sin(27^{\circ}) = \frac{\sqrt{8-2\sqrt{10-2\sqrt{5}}}}{4} = cos(63^{\circ})$$

$$sin(36^{\circ}) = \frac{\sqrt{10-2\sqrt{5}}}{4} = cos(54^{\circ})$$

$$sin(54^{\circ}) = \frac{1+\sqrt{5}}{4} = cos(36^{\circ})$$

$$sin(63^{\circ}) = \frac{\sqrt{8+2\sqrt{10-2\sqrt{5}}}}{4} = cos(27^{\circ})$$

$$sin(72^{\circ}) = \frac{\sqrt{10+2\sqrt{5}}}{4} = cos(18^{\circ})$$

$$sin(81^{\circ}) = \frac{\sqrt{8+2\sqrt{10+2\sqrt{5}}}}{4} = cos(9^{\circ})$$

To get started, we simply remind the reader of the angle addition formula for the $sine$ function and the double angle formula for the $cosine$ function.

$$sin(\alpha+\beta) = sin(\alpha)cos(\beta) + cos(\alpha)sin(\beta)$$

$$cos(2\cdot\theta) = 2\cdot cos^2(\theta) - 1$$

Now we make the remarkable observation that

$$sin(2\cdot 36^{\circ}) = sin(72^{\circ}) = sin(108^{\circ}) = sin(2\cdot 36^{\circ} + 36^{\circ})$$

Then we apply the double-angle formula for the $sine$ function on the left and also apply the angle addition formula for the $sine$ function on the right.

$$2\cdot sin(36^{\circ})\cdot cos(36^{\circ}) = sin(2\cdot 36^{\circ})\cdot cos(36^{\circ}) + cos(2\cdot 36^{\circ})\cdot sin(36^{\circ})$$

$$2\cdot sin(36^{\circ})\cdot cos(36^{\circ}) = 2\cdot sin(36^{\circ})\cdot cos^2(36^{\circ}) + cos(2\cdot 36^{\circ})\cdot sin(36^{\circ})$$

We know from drawing a unit circle picture that $sin(36^{\circ}) \neq 0$ so we may divide both sides of the last equation by this nonzero number and simplify that equation to:

$$2\cdot cos(36^{\circ}) = 2\cdot cos^2(36^{\circ}) + cos(2\cdot 36^{\circ})$$

$$2\cdot cos(36^{\circ}) = 2\cdot cos^2(36^{\circ}) + 2\cdot cos^2(36^{\circ}) - 1$$

$$0 = 4\cdot cos^2(36^{\circ}) - 2\cdot cos\big(36^{\circ}\big) - 1$$

This last equation is simply a quadratic equation so we write the answer for $cos(36^{\circ})$ by using the quadratic formula:

$$cos(36^{\circ}) = \frac{2\pm\sqrt{4+16}}{8} = \frac{2\pm\sqrt{20}}{8} = \frac{2\pm 2\sqrt{5}}{2\cdot 4} = \frac{1\pm\sqrt{5}}{4}$$

Finally we reason again by thinking of the unit circle picture for an angle of 36° that the sign of the final numerator must be a $+$ sign because $cos(36^{\circ})$ is a positive number. Thus we have established that

$$cos(36^{\circ}) = \frac{1+\sqrt{5}}{4}$$

Now it will be somewhat more straightforward to establish the remaining trigonometric function formula values shown at the beginning of this paper. For example, since $sin^2(36^{\circ}) + cos^2(36^{\circ}) = 1$ we find

$$sin^2(36^\circ) = 1 - \left(\frac{1+\sqrt{5}}{4}\right)^2 = \frac{16}{16} - \frac{1+2\sqrt{5}\ +5}{16} = \frac{16-(6+2\sqrt{5}\ )}{16} = \frac{10-2\sqrt{5}}{16}.$$

Now taking square roots and using the positive value only we have $sin(36^\circ) = \dfrac{\sqrt{10-2\sqrt{5}\ }}{4}$.

Next, using the half-angle formula for $sine$ with an appropriate sign we have

$$sin(18^\circ) = +\sqrt{\frac{1-cos(36^\circ)}{2}} = \sqrt{\frac{1}{2} - \frac{1+\sqrt{5}}{8}} = \sqrt{\frac{4-(1+\sqrt{5}\,)}{8}} =$$

$$\sqrt{\frac{3-\sqrt{5}}{8}} = \sqrt{\frac{6-2\sqrt{5}}{16}} = \frac{\sqrt{1-2\sqrt{5}\ +5}}{\sqrt{16}} = \frac{\sqrt{\left(-1+\sqrt{5}\right)^2}}{4} = \frac{-1+\sqrt{5}}{4}.$$

Now $sin(72^\circ) = 2 \cdot sin(36^\circ) \cdot cos(36^\circ) = 2 \cdot \dfrac{\sqrt{10-2\sqrt{5}\ }}{4} \cdot \dfrac{1+\sqrt{5}}{4} =$

$$\frac{\sqrt{10-2\sqrt{5}\ }}{2} \cdot \frac{\sqrt{\left(1+\sqrt{5}\right)^2}}{4} = \frac{\sqrt{\left(10-2\sqrt{5}\right)\cdot\left(6+2\sqrt{5}\right)}}{8} =$$

$$= \frac{\sqrt{60+8\sqrt{5}\ -20}}{8} = \frac{\sqrt{40+8\sqrt{5}\ }}{8} = \frac{\sqrt{4}\cdot\sqrt{10+2\sqrt{5}\ }}{8} = \frac{\sqrt{10+2\sqrt{5}}}{4}.$$

To establish the formulas for the multiples of 9° we start by applying the half-angle formulas for both $sine$ and $cosine$ functions.

$$sin(9^\circ) = sin\left(\frac{18^\circ}{2}\right) = +\sqrt{\frac{1-cos(18^\circ)}{2}} = \sqrt{\frac{1-\dfrac{\sqrt{10+2\sqrt{5}\ }}{4}}{2}} = \sqrt{\frac{4-\sqrt{10+2\sqrt{5}}}{8}}$$

$$= \sqrt{\frac{8-2\sqrt{10+2\sqrt{5}}}{16}} = \frac{\sqrt{8-2\sqrt{10+2\sqrt{5}}}}{4}.$$

Similarly, $\quad cos(9^\circ) = cos\left(\dfrac{18^\circ}{2}\right) = +\sqrt{\dfrac{1+cos(18^\circ)}{2}} = \sqrt{\dfrac{1+\dfrac{\sqrt{10+2\sqrt{5}}}{4}}{2}} =$

$$\sqrt{\frac{4+\sqrt{10+2\sqrt{5}}}{8}} = \sqrt{\frac{8+2\sqrt{10+2\sqrt{5}}}{16}} = \frac{\sqrt{8+2\sqrt{10+2\sqrt{5}}}}{4}.$$

Next, $sin(27^{\circ}) = sin\left(\frac{54^{\circ}}{2}\right) = +\sqrt{\frac{1-cos(54^{\circ})}{2}} = \sqrt{\frac{1-\frac{\sqrt{10-2\sqrt{5}}}{4}}{2}} =$

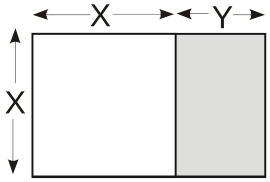
$$\sqrt{\frac{4-\sqrt{10-2\sqrt{5}}}{8}} = \sqrt{\frac{8-2\sqrt{10-2\sqrt{5}}}{16}} = \frac{\sqrt{8-2\sqrt{10-2\sqrt{5}}}}{4}.$$

Similarly, $cos(27^{\circ}) = cos\left(\frac{54^{\circ}}{2}\right) = +\sqrt{\frac{1+cos(54^{\circ})}{2}} = \sqrt{\frac{1+\frac{\sqrt{10-2\sqrt{5}}}{4}}{2}} =$

$$\sqrt{\frac{4+\sqrt{10-2\sqrt{5}}}{8}} = \sqrt{\frac{8+2\sqrt{10-2\sqrt{5}}}{16}} = \frac{\sqrt{8+2\sqrt{10-2\sqrt{5}}}}{4}.$$

The famous number called the Golden Ratio is a number that is related to first drawing a rectangle that is a little longer than it is wide such as the following:

Now we partition this rectangle into two sections, the one on the left is to be a perfect square, and the one on the right is to make a smaller rectangle that has the same aspect ratio as the original rectangle shown above.



In order to make the smaller rectangle on the right have the same aspect ratio $(length/height)$ as the original rectangle we must have x and y satisfy the fractional equation:

$$\frac{x+y}{x} = \frac{x}{y}$$

Now the width y can be chosen at random to conveniently be equal to a unit length. If we thus assume $y = 1$ then x must satisfy the equation that:

$$\frac{x+1}{x} = x$$

Finally, we rewrite this as the quadratic equation that: $0 = x^2 - x - 1$. We then apply the quadratic formula to get:

$$x = \frac{1 \pm \sqrt{1+4}}{2}$$

We can clearly see that x should not be negative so we take the $+$ sign and finally arrive at

$$x = \frac{1+\sqrt{5}}{2} \approx 1.61803$$

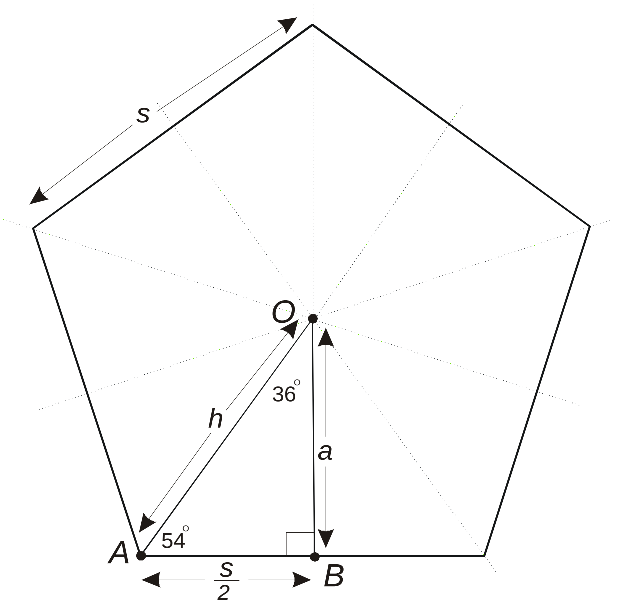
It is this famous number that is called the Golden Ratio. A rectangle that is about 1.618 times as wide as it is tall is said to have an aspect ratio that is the same as the Golden Ratio.

Previously we showed that $cos(36^{\circ}) = \dfrac{1+\sqrt{5}}{4} = \dfrac{\phi}{2}$ where $\phi =$ the Golden Ratio. Thus,

$$\phi = 2 \cdot cos(36^{\circ})$$

In the figure below we show a pentagon where each interior angle measures exactly 108°. Thus when we draw lines from the centroid to a side, making a $\perp$ angle with that side, and draw another line from the centroid to a vertex, the small right triangle $\triangle AOB$ that is formed is one in which the two acute angles measure exactly 54° and 36°.

We label one side of that small right triangle as a, representing a length known as the *apothem*. The shortest side in that triangle measures $\dfrac{s}{2}$ where s denotes the length that is a side of the pentagon. $AB = \dfrac{s}{2}$ and $OB = a$.



If we let $h =$ the length of the hypotenuse of $\triangle AOB$ then we can see that

$$cos(36^{\circ}) = \frac{a}{h} \quad \text{or} \quad a = h \cdot cos(36^{\circ}) = h \cdot \frac{1+\sqrt{5}}{4}$$

$$cos(54^{\circ}) = \frac{\frac{s}{2}}{h} \quad \text{or} \quad h = \frac{s}{2} \div cos(54^{\circ}) = \frac{s}{2} \cdot \frac{4}{\sqrt{10-2\sqrt{5}}} = \frac{2s}{\sqrt{10-2\sqrt{5}}}$$

The area of $\triangle AOB = \frac{1}{2} \cdot \frac{s}{2} \cdot a = \frac{s}{4} \cdot h \cdot \frac{1+\sqrt{5}}{4} = \frac{s}{4} \cdot \frac{2s}{\sqrt{10-2\sqrt{5}}} \cdot \frac{1+\sqrt{5}}{4}$

$= \dfrac{s^2\left(1+\sqrt{5}\right)}{8\sqrt{10-2\sqrt{5}}}$. Since there are 10 triangles like $\triangle AOB$ contained in the pentagon, the total area of the pentagon is given by:

$$\frac{10 \cdot s^2\left(1+\sqrt{5}\right)}{8\sqrt{10-2\sqrt{5}}} = \frac{5 \cdot s^2\left(1+\sqrt{5}\right)}{4\sqrt{10-2\sqrt{5}}} = \frac{5 \cdot s^2\left(1+\sqrt{5}\right)\sqrt{10-2\sqrt{5}}\,\left(10+2\sqrt{5}\right)}{4\sqrt{10-2\sqrt{5}}\,\sqrt{10-2\sqrt{5}}\,\left(10+2\sqrt{5}\right)}$$

(continued)

$$= \frac{5 \cdot s^2 \left(20 + 12\sqrt{5}\right)\sqrt{10 - 2\sqrt{5}}}{4\left(10 - 2\sqrt{5}\right)\left(10 + 2\sqrt{5}\right)} = \frac{5 \cdot s^2 \left(5 + 3\sqrt{5}\right)\sqrt{10 - 2\sqrt{5}}}{(100 - 20)}$$

$$= \frac{5 \cdot s^2 \left(5 + 3\sqrt{5}\right)\sqrt{10 - 2\sqrt{5}}}{80} = \frac{s^2 \sqrt{\left(5 + 3\sqrt{5}\right)^2}\,\sqrt{10 - 2\sqrt{5}}}{16}$$

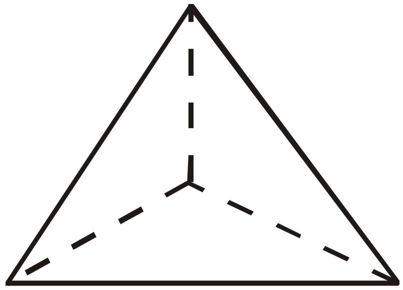
$$= \frac{s^2 \sqrt{\left(25 + 30\sqrt{5} + 45\right)}\,\sqrt{10 - 2\sqrt{5}}}{16} = \frac{s^2 \sqrt{\left(70 + 30\sqrt{5}\right) \cdot \left(10 - 2\sqrt{5}\right)}}{16}$$

$$= \frac{s^2 \sqrt{700 + 300\sqrt{5} - 140\sqrt{5} - 300}}{16} = \frac{s^2 \sqrt{400 + 160\sqrt{5}}}{16} = \frac{s^2 \sqrt{16}\,\sqrt{25 + 10\sqrt{5}}}{4 \cdot 4}$$

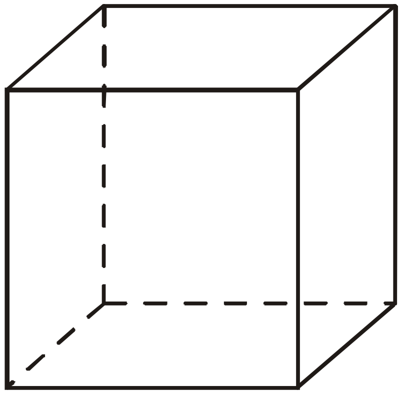
$$= \frac{s^2 \sqrt{25 + 10\sqrt{5}}}{4}.$$

Some Polyhedra Information

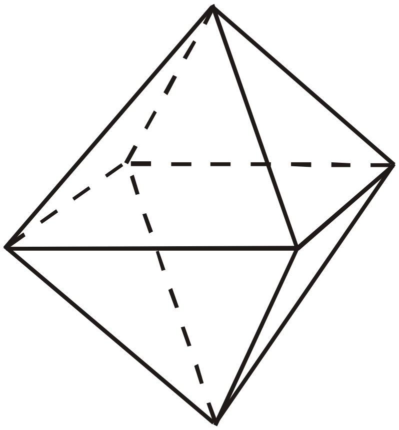
In classic geometry there are five famous 3-dimensional regular solids that are pictured below.



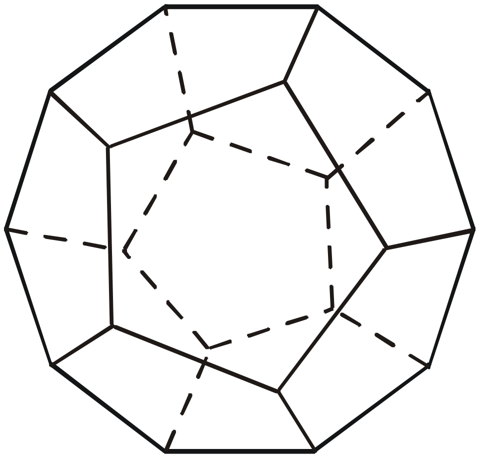
Tetrahedron



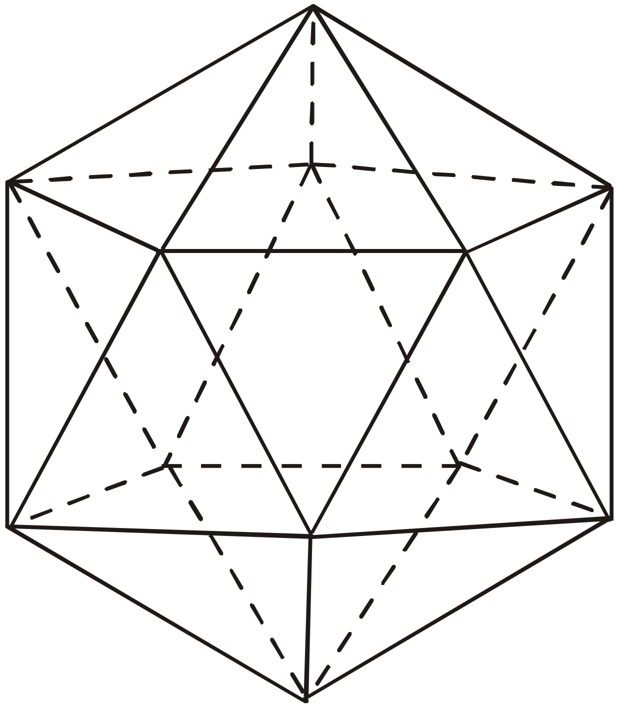
Cube (Hexahedron)



Octahedron



Dodecahedron



Icosahedron

In the two tables that follow on the next two pages, the variable s denotes the length of a side in any of the polyhedra. In general, the total surface area and volume and the CircumRadius and the InRadius all depend on s, the length of a side. In turn, s, determines the overall size of the polyhedra. Note that s can be scaled either up or down to make any particular size. Strangely enough, the dihedral angle, which is the angle between the two planes containing adjacent faces, does not depend at all on s. Think of s as the length of each edge of the polyhedra.

The CircumRadius is the radius of the smallest sphere that surrounds or contains the regular 3D polyhedra. That sphere can also be called the Circumscribed sphere. The InRadius is the radius of the largest sphere that is contained in the interior of the polyhedra. That sphere can be called the Inscribed sphere. Note that the vertices of the polyhedra all lie on the Circumscribed sphere. The plane faces of the polyhedra are tangent to the Inscribed sphere, with the points of tangency at the centroid point of each face. The Inscribed sphere does not contain any vertices. The InRadius is always smaller than the CircumRadius.

$s =$ the length of a side or edge

Polyhedra Name	Volume	Surface Area	# Vertices	# Edges	# Faces
Tetrahedron	$\dfrac{\sqrt{2}}{12} \cdot s^3$ $\approx 0.117851130198 \cdot s^3$	$\sqrt{3} \cdot s^2$ $\approx 1.73205080757 \cdot s^2$	4	6	4
Cube (Hexahedron)	s^3	$6s^2$	8	12	6
Octahedron	$\dfrac{\sqrt{2}}{3} \cdot s^3$ $\approx 0.47140452079 \cdot s^3$	$2\sqrt{3} \cdot s^2$ $\approx 3.46410161514 \cdot s^2$	6	12	8
Dodecahedron	$\dfrac{(15 + 7\sqrt{5})}{4} \cdot s^3$ $\approx 7.66311896062 \cdot s^3$	$3\sqrt{25 + 10\sqrt{5}} \cdot s^2$ $\approx 20.6457288071 \cdot s^2$	20	30	12
Icosahedron	$\dfrac{5(3 + \sqrt{5})}{12} \cdot s^3$ $\approx 2.18169499062 \cdot s^3$	$5\sqrt{3} \cdot s^2$ $\approx 8.66025403785 \cdot s^2$	12	30	20

The values in the next table can be established by analyzing the corresponding polyhedra in detail. Proofs of all these facts can be found in the paper titled ***Establishing Regular Polyhedra Formula*** that is on the author's web site: **homepage.smc.edu\kennedy_john**.

Polyhedra Name	Dihedral Angle	CircumRadius	InRadius
Tetrahedron	$tan^{-1}\left(2\sqrt{2}\right) \approx 70.528779^{\circ}$	$\frac{\sqrt{6}\cdot s}{4}$ $\approx 0.612372435695 \cdot s$	$\frac{\sqrt{6}\cdot s}{12}$ $\approx 0.204124145232 \cdot s$
Cube (Hexahedron)	$sin^{-1}(1) = 90^{\circ}$	$\frac{\sqrt{3}\cdot s}{2}$ $\approx 0.866025403785 \cdot s$	$\frac{s}{2}$ $= 0.5 \cdot s$
Octahedron	$2 \cdot tan^{-1}(\sqrt{2}) \approx 109.47122^{\circ}$	$\frac{s}{\sqrt{2}}$ $\approx 0.707106781188 \cdot s$	$\frac{s}{\sqrt{6}}$ $\approx 0.408248290464 \cdot s$
Dodecahedron	$2 \cdot tan^{-1}\left(\frac{1+\sqrt{5}}{2}\right) \approx 116.56505^{\circ}$	$\frac{\sqrt{3}\cdot(1+\sqrt{5})s}{4}$ $\approx 1.40125853844 \cdot s$	$\frac{\sqrt{250+110\sqrt{5}}}{20} \cdot s$ $\approx 1.11351636441 \cdot s$
Icosahedron	$2 \cdot tan^{-1}\left(\frac{3+\sqrt{5}}{2}\right) \approx 138.189685^{\circ}$	$\frac{\sqrt{10+2\sqrt{5}}\cdot s}{4}$ $\approx 0.951056516295 \cdot s$	$\frac{\sqrt{42+18\sqrt{5}}}{12} \cdot s$ $\approx 0.755761314076 \cdot s$

$s =$ the length of a side or edge

In the case of the dodecahedron we can see that the dihedral angle is closely associated with the Golden Ratio. In fact, that angle is $2 \cdot tan^{-1}(\phi)$ where ϕ denotes the Golden Ratio. Also associated with the dodecahedron is the CircumRadius that is $\frac{\sqrt{3}}{2} \cdot \phi \cdot s$ where s denotes the length of a side or an edge.

In the case of the icosahedron the dihedral angle is $2 \cdot tan^{-1}(1+\phi)$. We also see the CircumRadius of the icosahedron is the same as $sin\left(72^{\circ}\right) \cdot s$ where s is the length of a side or an edge. $\frac{\sqrt{10+2\sqrt{5}}}{4} = sin(72^{\circ}) = cos\left(18^{\circ}\right)$. We can go even further and write this last value as:

$$\frac{\sqrt{10+2\sqrt{5}}}{4} = \frac{\sqrt{1+2\sqrt{5}+5+4}}{4} = \frac{\sqrt{\left(1+\sqrt{5}\right)^2+4}}{4} = \frac{\sqrt{4}\cdot\sqrt{\frac{\left(1+\sqrt{5}\right)^2}{4}+1}}{4} = \frac{\sqrt{\phi^2+1}}{2}.$$