

Solving Cubic Equations

by

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Solving Cubic Equations

Assume the original cubic equation is of the form:

$$ax^3 + bx^2 + cx + d = 0 \quad \text{where } a \neq 0.$$

We can factor out the leading coefficient a and re-write the polynomial equation in the form:

$$a\left[x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a}\right] = 0.$$

Since $a \neq 0$ we can divide by a and thus assume that our cubic is a monic polynomial of the form:

$$x^3 + Ax^2 + Bx + C = 0$$

where $A = \frac{b}{a}$ and $B = \frac{c}{a}$ and $C = \frac{d}{a}$.

Next we will make the change of variable $y = x + \frac{A}{3}$. We do this to derive a new cubic equation in which the x^2 term will be missing. It is not obvious why or how this works so we must show the details.

Note that $x = y - \frac{A}{3}$ so that the above equation becomes:

$$\left(y - \frac{A}{3}\right)^3 + A\left(y - \frac{A}{3}\right)^2 + B\left(y - \frac{A}{3}\right) + C = 0$$

$$y^3 - 3 \cdot y^2\left(\frac{A}{3}\right) + 3y\left(\frac{A}{3}\right)^2 - \left(\frac{A}{3}\right)^3 + Ay^2 - A\frac{2A}{3}y + A\left(\frac{A}{3}\right)^2 + By - \frac{AB}{3} + C = 0$$

$$y^3 - Ay^2 + y\frac{A^2}{3} - \frac{A^3}{27} + Ay^2 - \frac{2A^2}{3}y + \frac{A^3}{9} + By - \frac{AB}{3} + C = 0$$

$$y^3 + y\frac{A^2}{3} - \frac{A^3}{27} - \frac{2A^2}{3}y + \frac{A^3}{9} + By - \frac{AB}{3} + C = 0$$

This shows that we can reduce our cubic equation to one that is of the form:

$$y^3 + Ry + S = 0$$

In turn, we temporarily switch back to using the x variable and re-write this last equation in the form $x^3 = Mx + N$. This represents the intersection of the line $y = Mx + N$ with the graph of $y = x^3$. This last form is intellectually interesting where we consider the two cases of both positively and negatively sloped lines. See Figure 1 on the next page. A negatively sloped line will intersect $y = x^3$ in only one point while a positively sloped line *may* intersect in three points, but will intersect in at least one point.

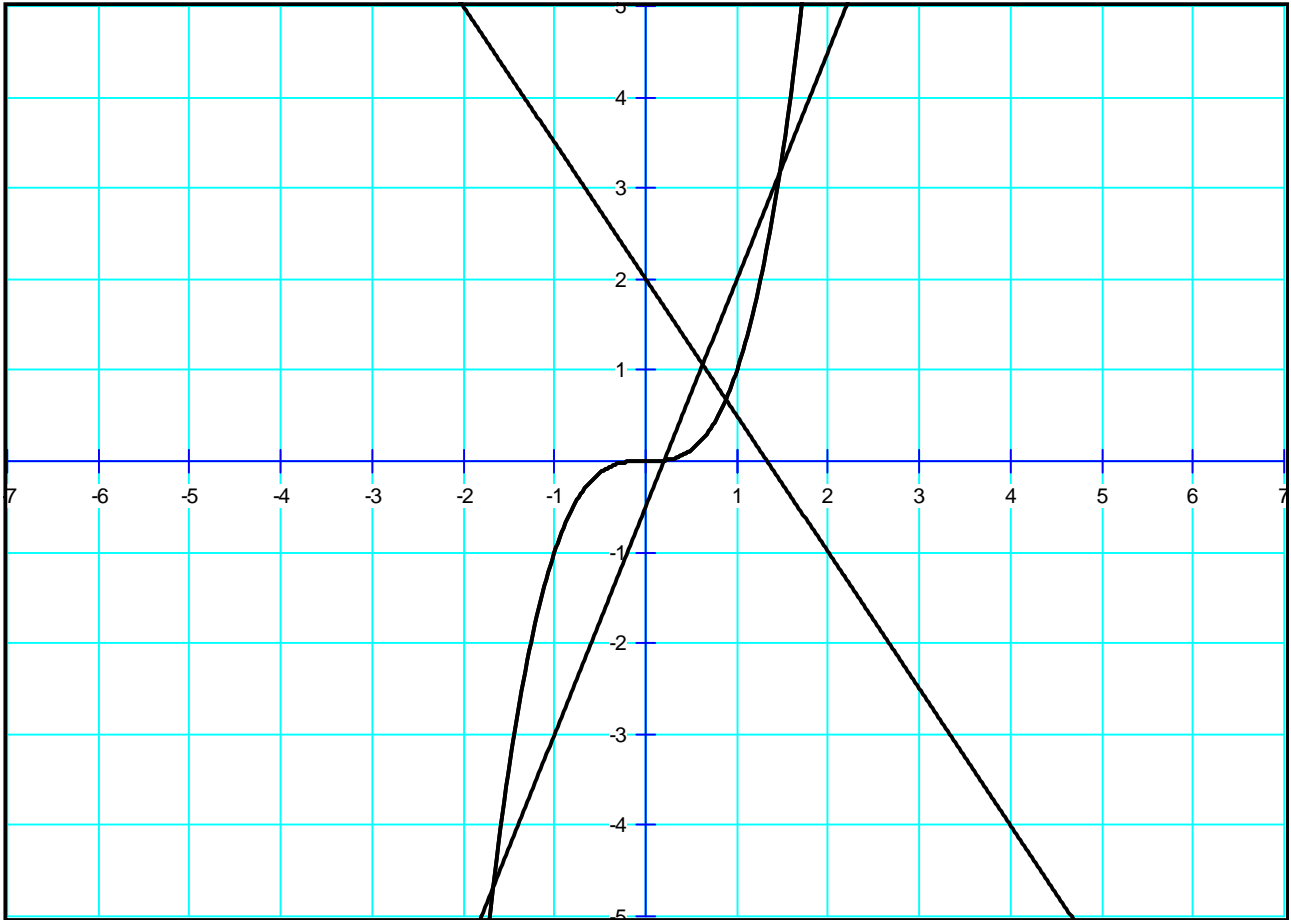


Figure 1 The graph of $y = x^3$ intersected with two lines.

In any case we switch back to using y and re-write the last equation in the more specific cubic polynomial form:

$$y^3 = 3py + 2q \tag{*}$$

Now we make an unusual assumption that turns out to make a solution easy to find. We assume any solution to the (*) equation can be written in the form

$$y = s + t$$

where $s^3 + t^3 = 2q$ and where $st = p$.

We will substitute $y = s + t$ in the above (*) equation and show that it leads to a solution provided $st = p$ and $s^3 + t^3 = 2q$. Here we go with the simultaneous substitutions on both sides of the (*) equation.

$$\begin{aligned}
(s+t)^3 &\stackrel{?}{=} 3p(s+t) + 2q \\
s^3 + 3s^2t + 3st^2 + t^3 &\stackrel{?}{=} 3ps + 3pt + 2q \\
3s^2t + 3st^2 &\stackrel{?}{=} 3ps + 3pt \\
3sp + 3pt &\not\equiv 3ps + 3pt
\end{aligned}$$

At last we see that $s+t$ is indeed a solution to the above (*) cubic equation.

Now we will transform the (*) cubic equation one last time by letting $y = s + \frac{p}{s}$.

$$\begin{aligned}
\left(s + \frac{p}{s}\right)^3 &= 3p\left(s + \frac{p}{s}\right) + 2q \\
s^3 + 3s^2\left(\frac{p}{s}\right) + 3s\left(\frac{p}{s}\right)^2 + \left(\frac{p}{s}\right)^3 &= 3ps + 3\frac{p^2}{s} + 2q \\
s^3 + 3ps + 3\frac{p^2}{s} + \frac{p^3}{s^3} &= 3ps + 3\frac{p^2}{s} + 2q \\
s^6 + 3ps^4 + 3p^2s^2 + p^3 &= 3ps^4 + 3p^2s^2 + 2qs^3 \\
s^6 + p^3 &= 2qs^3 \\
(s^3)^2 - 2q(s^3) + p^3 &= 0
\end{aligned}$$

This last equation is nothing more than a quadratic equation in s^3 so we may solve it using the quadratic formula.

$$s^3 = \frac{2q \pm \sqrt{4q^2 - 4p^3}}{2} = \frac{2q \pm 2\sqrt{q^2 - p^3}}{2} = q \pm \sqrt{q^2 - p^3}$$

Finally we see the two potential answers for s , each of which only depends on the p and q coefficients.

$$s = \sqrt[3]{q + \sqrt{q^2 - p^3}} \quad \text{or} \quad s = \sqrt[3]{q - \sqrt{q^2 - p^3}}$$

Now once we know s we can find t since $s^3 + t^3 = 2q$. If we assume $s = \sqrt[3]{q + \sqrt{q^2 - p^3}}$ then $s^3 = q + \sqrt{q^2 - p^3}$ so we know $2q - \{q + \sqrt{q^2 - p^3}\} = t^3$ or $t = \sqrt[3]{q - \sqrt{q^2 - p^3}}$. Similarly, if $s = \sqrt[3]{q - \sqrt{q^2 - p^3}}$ then we can show that $t^3 = 2q - \{q - \sqrt{q^2 - p^3}\}$ or $t = \sqrt[3]{q + \sqrt{q^2 - p^3}}$. The end result is a solution for y :

$$y = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}} \quad (**)$$

or

$y = \sqrt[3]{q - \sqrt{q^2 - p^3}} + \sqrt[3]{q + \sqrt{q^2 - p^3}}$ which is the same as (**). Again, this is strange evidence of symmetry because we get the same answer for y no matter which choice of s we make. In other words, the two possible answers for s lead to the same unique answer for y given by the (**) equation.

To actually finish solving for x , we write $x = y - \frac{A}{3} = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}} - \frac{A}{3}$.

In practice, it is best to make the changes in variables as we have indicated and keep transforming and making equivalent equations. We can see that when $q^2 - p^3 < 0$ then the above (**) equation leads to two complex numbers that are added together. However, those complex numbers are complex conjugates of one another and as we will see, their sum causes the imaginary parts to cancel and their real parts to double, resulting in a real answer for y .

We will work a couple of numerical examples that will show how to handle the details.

Our first equation is: $x^3 - 15x - 4 = 0$. When we re-arrange this to get $x^3 = 15x + 4$ we can think where does the graph of the line $y = 15x + 4$ intersect the graph of $y = x^3$?

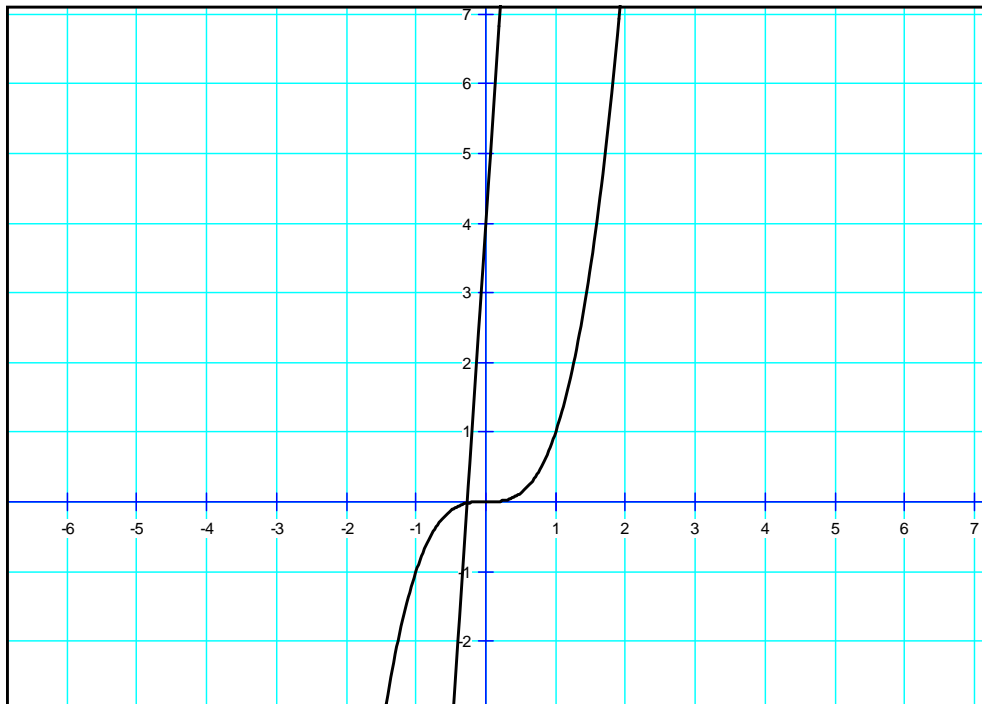


Figure 2. An intersection point of the line $y = 15x + 4$ with the cubic $y = x^3$.

In the Figure 2 we can see the point of intersection where x should be a small negative number between -1 and 0 , but closer to 0 . We might estimate x is near $-\frac{1}{3}$.

Now we re-write the original cubic equation in the special form:

$$x^3 = 3 \cdot 5x + 2 \cdot 2$$

Now we can see that $p = 5$ and $q = 2$. Then,

$$\begin{aligned} x &= \sqrt[3]{2 + \sqrt{4 - 125}} + \sqrt[3]{2 - \sqrt{4 - 125}} \\ &= \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} \\ &= \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i} \end{aligned}$$

Now it is easiest to find the cube roots of complex numbers by writing those numbers in polar form. $a + bi = re^{i\theta}$ where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$. Note that $e^{i\theta}$ is always a point on the unit circle by Euler's Identity. $e^{i\theta} = \cos(\theta) + i \cdot \sin(\theta)$. In polar form, the cube root of a complex number has an angle that is $\frac{1}{3}$ of the angle for the original complex number and it has a radius that is the cube root of the original radius. These facts are easily derived from DeMoivre's Theorem. In order to derive exact values of *cosine* we would like to have a $\frac{1}{3}$ -angle formula for the *cosine* function, but this is difficult to come by.

The complex number $2 + 11i = \sqrt{125}e^{i \cdot \tan^{-1}\left(\frac{11}{2}\right)}$ so its cube root is $\sqrt[3]{\sqrt{125}} e^{i \cdot \frac{1}{3} \tan^{-1}\left(\frac{11}{2}\right)} = \sqrt[3]{\sqrt{125}} \left\{ \cos\left(\frac{1}{3} \tan^{-1}\left(\frac{11}{2}\right)\right) + i \cdot \sin\left(\frac{1}{3} \tan^{-1}\left(\frac{11}{2}\right)\right) \right\}$.

Similarly, $\sqrt[3]{2 - 11i} = \sqrt[3]{\sqrt{125}} \left\{ \cos\left(\frac{1}{3} \tan^{-1}\left(\frac{-11}{2}\right)\right) + i \cdot \sin\left(\frac{1}{3} \tan^{-1}\left(\frac{-11}{2}\right)\right) \right\} = \sqrt[3]{\sqrt{125}} \left\{ \cos\left(\frac{1}{3} \tan^{-1}\left(\frac{11}{2}\right)\right) - i \cdot \sin\left(\frac{1}{3} \tan^{-1}\left(\frac{11}{2}\right)\right) \right\}$.

Thus when we add these two cube roots the imaginary parts cancel and the real parts double. At this point we discover that $\theta = \tan^{-1}\left(\frac{11}{2}\right)$ or $\tan(\theta) = \frac{11}{2}$. We can easily find that $\cos(\theta) = \frac{2}{5\sqrt{5}}$ and $\sin(\theta) = \frac{11}{5\sqrt{5}}$. However, it is not easy to find the exact cosine value of $\frac{1}{3}$ of this angle. With a calculator we find $\cos^2\left(\frac{\theta}{3}\right) = 0.8$ so that $\cos\left(\frac{\theta}{3}\right) = \sqrt{0.8}$.

Now we can calculate x exactly! $x = 2 \cdot \sqrt[3]{\sqrt{125}} \cos\left(\frac{1}{3} \tan^{-1}\left(\frac{11}{2}\right)\right) = 2 \cdot 125^{\frac{1}{6}} \cdot \cos\left(\frac{1}{3} \tan^{-1}\left(\frac{11}{2}\right)\right) = 2\sqrt{5} \cdot \sqrt{0.8} = 2\sqrt{5(0.8)} = 2\sqrt{4} = 2 \cdot 2 = 4$.

Clearly we can see that $x = 4$ satisfies the equation: $x^3 = 15x + 4$. Thus we have found one real solution to the cubic equation $x^3 - 15x - 4 = 0$. By Synthetic Division we can deflate this polynomial to derive the quadratic factor.

$$\begin{array}{r|rrrr} & & & & 4 \\ 1 & 1 & 0 & -15 & -4 \\ & & 4 & 16 & 4 \\ \hline & 1 & 4 & 1 & 0 \end{array}$$

Now we can find the other two real roots by solving the quadratic: $x^2 + 4x + 1 = 0$. The other two solutions are thus

$$x = \frac{-4 \pm \sqrt{16 - 4}}{2} = \frac{-4 \pm \sqrt{12}}{2} = \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$$

We should note that finding these other two real solutions would not have been easy had we not found the one real solution first that required handling complex numbers.

Moreover, since $\sqrt{3} \approx 1.732$ we can reason that the root we saw in Figure 2 is really the number $-2 + \sqrt{3} \approx -0.268$. The other irrational root $-2 - \sqrt{3}$ is nowhere apparent in Figure 2, nor is the root $x = 4$ apparent in Figure 2.

Here's another numerical example that will lead us through slightly more messy details involving fractions and radicals with fractions.

Solve the equation: $2x^3 + 10x^2 + 8x - 4 = 0$.

This first becomes $x^3 + 5x^2 + 4x - 2 = 0$ after we divide by 2.

Now we make a change of variable and let $x = y - \frac{5}{3}$.

$$\left(y - \frac{5}{3}\right)^3 + 5\left(y - \frac{5}{3}\right)^2 + 4\left(y - \frac{5}{3}\right) - 2 = 0.$$

$$y^3 - 3y^2\left(\frac{5}{3}\right) + 3y\left(\frac{5}{3}\right)^2 - \left(\frac{5}{3}\right)^3 + 5y^2 - \frac{50}{3}y + \frac{125}{9} + 4y - \frac{20}{3} - 2 = 0$$

$$y^3 - 5y^2 + \frac{25}{3}y - \frac{125}{27} + 5y^2 - \frac{50}{3}y + \frac{125}{9} + 4y - \frac{20}{3} - 2 = 0$$

$$y^3 - \frac{25}{3}y + 4y = \frac{125}{27} - \frac{125}{9} + \frac{20}{3} + 2$$

$$y^3 = \frac{25}{3}y - \frac{12}{3}y + \frac{125}{27} - \frac{375}{27} + \frac{180}{27} + \frac{54}{27}$$

$$y^3 = \frac{13}{3}y + \frac{-16}{27}$$

Now we determine the p and q values by writing this last equation in the form:

$$y^3 = 3 \cdot \frac{13}{9}y + 2\left(\frac{-8}{27}\right)$$

Then $p = \frac{13}{9}$ and $q = \frac{-8}{27}$.

Finally we calculate $y = \sqrt[3]{\frac{-8}{27} + \sqrt{\frac{64}{729} - \frac{2197}{729}}} + \sqrt[3]{\frac{-8}{27} - \sqrt{\frac{64}{729} - \frac{2197}{729}}}$

$$y = \sqrt[3]{\frac{-8}{27} + \sqrt{\frac{-2133}{729}}} + \sqrt[3]{\frac{-8}{27} - \sqrt{\frac{-2133}{729}}}$$

$$y = \sqrt[3]{\frac{-8}{27} + \sqrt{\frac{-79}{27}}} + \sqrt[3]{\frac{-8}{27} - \sqrt{\frac{-79}{27}}}$$

$$y = \sqrt[3]{\frac{-8}{27} + \frac{1}{3}\sqrt{\frac{79}{3}}i} + \sqrt[3]{\frac{-8}{27} - \frac{1}{3}\sqrt{\frac{79}{3}}i}$$

$$y = \sqrt[3]{\frac{-8}{27} + \frac{\sqrt{237}}{9}i} + \sqrt[3]{\frac{-8}{27} - \frac{\sqrt{237}}{9}i}$$

$$y = \sqrt[3]{\frac{-8}{27} + \frac{3\sqrt{237}}{27}i} + \sqrt[3]{\frac{-8}{27} - \frac{3\sqrt{237}}{27}i}$$

Again we have a sum of cube roots of complex number conjugates so the real parts double and the imaginary parts cancel. The radius of the complex number $\frac{-8}{27} + \frac{3\sqrt{237}}{27}i$ is :

$$\sqrt{\frac{64}{729} + \frac{9 \cdot 237}{729}} = \sqrt{\frac{64}{729} + \frac{2133}{729}} = \sqrt{\frac{2197}{729}} = \frac{13\sqrt{13}}{27}.$$

The polar angle of $\frac{-8}{27} + \frac{3\sqrt{237}}{27}i$ is : $\theta = \tan^{-1}\left(\frac{\frac{3\sqrt{237}}{27}}{\frac{-8}{27}}\right) = \tan^{-1}\left(\frac{3\sqrt{237}}{-8}\right)$. However, we can see the

number lies in the second quadrant. In degrees the θ -angle value for the complex number $\frac{-8}{27} + \frac{3\sqrt{237}}{27}i$ is approximately $\theta \approx -80.1728173731^\circ + 180^\circ = 99.8271826269^\circ$.

$\cos\left(\frac{\theta}{3}\right) = 0.836039871261$. Now we can try to calculate y .

$$y = 2 \cdot \sqrt[3]{\frac{13\sqrt{13}}{27}} \cdot \cos\left(\frac{1}{3}\tan^{-1}\left(\frac{3\sqrt{237}}{-8}\right)\right) = 2 \cdot 1.20185042515 \cdot 0.836039871261 \approx 2.00958974944.$$

Finally, $x = y - \frac{5}{3} \approx 0.34292308277$.

Alternative Methods Using Trigonometric and Hyperbolic Functions

One alternative method for solving any cubic equation in the form $x^3 + ax + b = 0$ is to exploit the fundamental trigonometric identity that $4 \cdot \cos^3(\theta) - 3 \cdot \cos(\theta) = \cos(3 \cdot \theta)$. To do this we will temporarily assume $a < 0$ and we will later make another assumption regarding the size of b in relation to the size of a .

To get started we replace x in the original equation with the expression $k \cdot \cos(\theta)$ where the constant k is yet to be determined.

$$k^3 \cos^3(\theta) + ak \cos(\theta) + b = 0$$

Now multiply through by $\frac{4}{k^3}$ to get the equation in the form:

$$4 \cos^3(\theta) + \frac{4a}{k^2} \cos(\theta) = \frac{-4b}{k^3}$$

Now we choose k so that the coefficient on the second term equals -3 . $\frac{4a}{k^2} = -3$.

Choose $k = \sqrt{\frac{4a}{-3}}$. When $a < 0$ then k is well-defined. Then we have :

$$4 \cos^3(\theta) - 3 \cos(\theta) = \frac{-4b}{k^3}$$

Now we further assume $-1 \leq \frac{-4b}{k^3} \leq +1$. Then we can continue by setting $\cos(3\theta) = \frac{-4b}{k^3}$ and finding all the possible answers for the angle θ . The three answers to the original cubic equation are simply $x = k \cos(\theta)$ where we note that even though there may be infinitely many answers for θ all but three of those repeat the same $\cos(\theta)$ values.

We now give an example using this new method.

Solve the equation $x^3 = 6x + 4$.

$$x^3 - 6x = 4$$

$$k^3 \cos^3(\theta) - 6k \cos(\theta) = 4$$

$$4 \cos^3(\theta) - \frac{4 \cdot 6}{k^2} \cos(\theta) = \frac{4 \cdot 4}{k^3}$$

Now we choose k such that $\frac{24}{k^2} = 3$. In other words, choose $k = \sqrt{8}$.

$$4\cos^3(\theta) - 3\cos(\theta) = \frac{16}{8\sqrt{8}}$$

$$4\cos^3(\theta) - 3\cos(\theta) = \frac{16\sqrt{8}}{64} = \frac{\sqrt{8}}{4} = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2}.$$

$$4\cos^3(\theta) - 3\cos(\theta) = \frac{\sqrt{2}}{2}.$$

Now we set $\cos(3\theta) = \frac{\sqrt{2}}{2}$ and try to determine the exact value of $\cos(\theta)$.

Then we know $3\theta = \frac{\pi}{4} \pm 2\pi n$ or $3\theta = \frac{7\pi}{4} \pm 2\pi n$.

$$\theta = \frac{\pi}{12} \pm \frac{2\pi}{3}n \text{ or } \theta = \frac{7\pi}{12} \pm \frac{2\pi}{3}n.$$

In terms of degrees we find there are three distinct cosine values.

We can write $x = \sqrt{8}\cos(15^\circ)$ or $x = \sqrt{8}\cos(135^\circ)$ or $x = \sqrt{8}\cos(225^\circ)$.

It can be shown that $\cos(15^\circ) = \cos(45^\circ - 30^\circ) = \cos(45^\circ)\cos(30^\circ) + \sin(45^\circ)\sin(30^\circ) = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4} \approx 0.965925826$.

Thus for our first x we have $x = \sqrt{8} \cdot \frac{\sqrt{6}+\sqrt{2}}{4} = \frac{\sqrt{48}+\sqrt{16}}{4} = \frac{4\sqrt{3}+4}{4} = \sqrt{3} + 1 \approx 2.73205080757$.

For our second x it can be shown $\cos(105^\circ) = \cos(135^\circ - 30^\circ) = \cos(135^\circ)\cos(30^\circ) + \sin(135^\circ)\sin(30^\circ) = \frac{-\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{-\sqrt{6}+\sqrt{2}}{4} \approx -0.258819045102$.

Thus for our second x we have

$$x = \sqrt{8} \cdot \frac{-\sqrt{6}+\sqrt{2}}{4} = \frac{-\sqrt{48}+\sqrt{16}}{4} = \frac{-4\sqrt{3}+4}{4} = -\sqrt{3} + 1 \approx -0.73205080757.$$

Our third x value is the easiest. $\cos(225^\circ) = -\frac{\sqrt{2}}{2}$ so $x = \sqrt{8} \cdot \left(\frac{-\sqrt{2}}{2}\right) = \frac{-\sqrt{16}}{2} = \frac{-4}{2} = -2$.

Now we come back and explain how to handle the case where $a > 0$ in the cubic equation $x^3 + ax + b = 0$. In this case we can exploit the hyperbolic trigonometric identity that says $4 \cdot \sinh^3(\theta) + 3 \cdot \sinh(\theta) = \sinh(3\theta)$. If you understood the case for using $\cos(\theta)$ above then using the hyperbolic sine function will be just as simple for you to apply. Just mimic the same steps but substitute $x = k \cdot \sinh(\theta)$ and later decide how to set the value for k .

The other cases we couldn't handle using $\cos(\theta)$ is when $a < 0$ but the fraction $\frac{-4b}{k^3}$ is such that $\frac{-4b}{k^3} < -1$ or $\frac{-4b}{k^3} > +1$. In these cases we can use the hyperbolic cosine function and exploit the identity that says $4 \cdot \cosh^3(\theta) - 3 \cdot \cosh(\theta) = \cosh(3\theta)$. Again, if you followed how to use the regular cosine function then using $\cosh(\theta)$ should be easy.

Historically the Italian Renaissance mathematician del Ferro published his results in 1515. Thus we can understand most of this material is several hundred years old. Other Italian Renaissance mathematicians who contributed to the results discussed in this paper were Tartaglia and Cardano (1545) and Bombelli (1572). Francois Viète published his results in 1591. Solving cubic equations was responsible for the development of algebra in Europe which was further stimulated by the work to find solutions to polynomial equations of higher degrees. Later both Euler and Gauss contributed to popularizing the use of complex numbers.

After reading this paper you may wish to read another paper by the same author that is titled **The Pure Cubic Polynomial and Intersections With Linear Functions.**

Just visit http://homepage.smc.edu/kennedy_john