Lower Riemann Sum

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

$$x_i = x_0 + i \cdot \Delta x = 0 + i \cdot \frac{3}{n} = \frac{3i}{n}$$

$$\text{Exact Area } = \lim_{n\to\infty} \left\{ \sum_{i=1}^{n} f(x_i)\Delta x \right\}$$

$$= \lim_{n\to\infty} \left\{ \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n} \right\}$$

$$= \lim_{n\to\infty} \left\{ \sum_{i=1}^{n} \left[9 - \left(\frac{3i}{n}\right)^2\right]\frac{3}{n} \right\}$$

$$= \lim_{n\to\infty} \left\{ \frac{3}{n}\sum_{i=1}^{n} \left[9 - \frac{9i^2}{n^2}\right] \right\}$$

$$= \lim_{n\to\infty} \frac{3}{n}\left\{ \sum_{i=1}^{n} 9 - \sum_{i=1}^{n} \frac{9i^2}{n^2} \right\}$$

$$= \lim_{n\to\infty} \frac{3}{n}\left\{ 9n - \frac{9}{n^2}\sum_{i=1}^{n} i^2 \right\}$$

$$= \lim_{n\to\infty} \frac{3}{n}\left\{ 9n - \frac{9}{n^2}\left[\frac{n(n+1)(2n+1)}{6}\right] \right\}$$

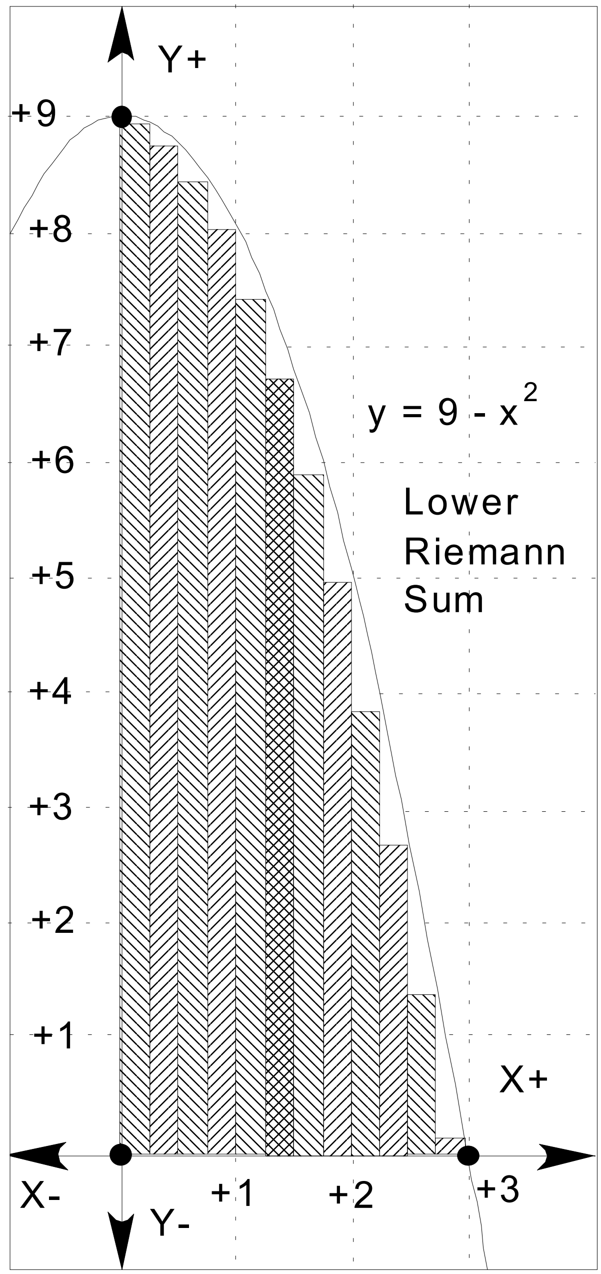
$$= \lim_{n\to\infty} \frac{3}{n}\left\{ 9n - \frac{9}{n^2}\left[\frac{2n^3+3n^2+n}{6}\right] \right\}$$

$$= \lim_{n\to\infty} \frac{3}{n}\left\{ 9n - 3n - \frac{9}{2} - \frac{3}{2n} \right\}$$

$$= \lim_{n\to\infty} \left\{ 27 - 9 - \frac{27}{2n} - \frac{9}{2n^2} \right\}$$

$$= \lim_{n\to\infty} \left\{ 18 - \frac{27}{2n} - \frac{9}{2n^2} \right\}$$

$$= 18$$



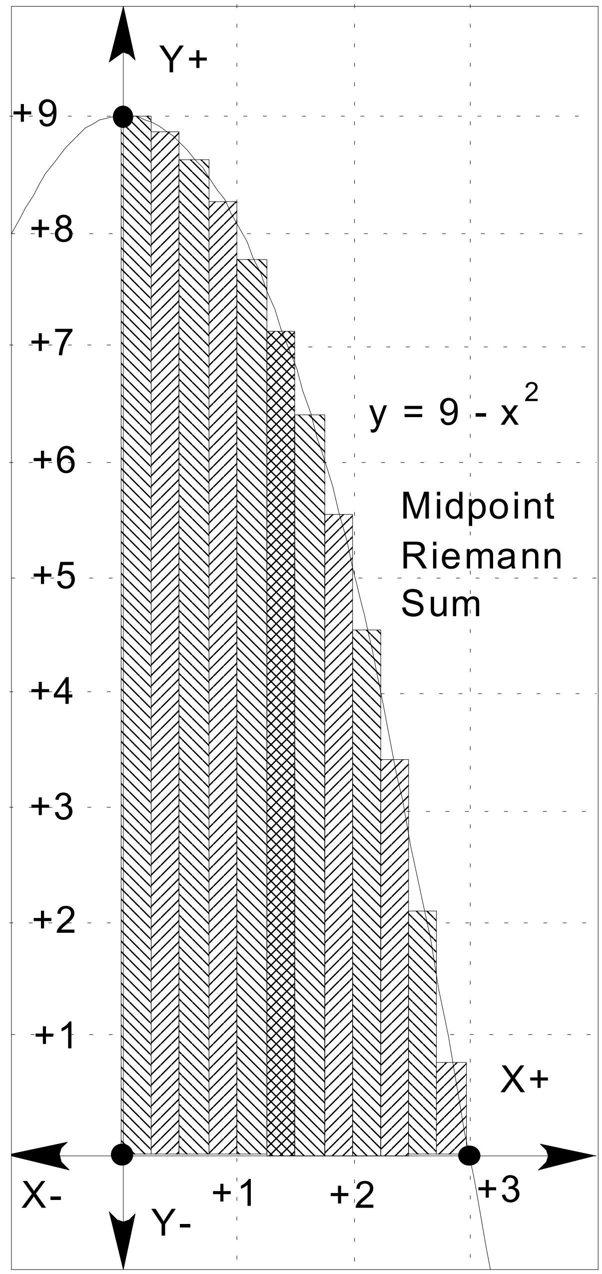
In the next to the last step, the expression inside the braces is the exact area of the n lower rectangles.

Midpoint Riemann Sum

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

$$x_i = x_0 + \frac{\Delta x}{2} + (i-1)\cdot\Delta x = 0 + \left(\frac{1}{2} + i - 1\right)\Delta x$$

$$= \left(i - \frac{1}{2}\right)\cdot\frac{3}{n} = \frac{2i-1}{2}\cdot\frac{3}{n} = \frac{6i-3}{2n}$$



$$\text{Exact Area } = \lim_{n\to\infty}\left\{\sum_{i=1}^{n} f(x_i)\Delta x\right\}$$

$$= \lim_{n\to\infty}\left\{\sum_{i=1}^{n} f\left(\frac{6i-3}{2n}\right)\frac{3}{n}\right\}$$

$$= \lim_{n\to\infty}\left\{\sum_{i=1}^{n}\left[9 - \left(\frac{6i-3}{2n}\right)^2\right]\frac{3}{n}\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{3}{n}\sum_{i=1}^{n}\left[9 - \frac{36i^2-36i+9}{4n^2}\right]\right\}$$

$$= \lim_{n\to\infty}\frac{3}{n}\left\{\sum_{i=1}^{n}9 - \sum_{i=1}^{n}\frac{36i^2}{4n^2} + \sum_{i=1}^{n}\frac{36i}{4n^2} - \sum_{i=1}^{n}\frac{9}{4n^2}\right\}$$

$$= \lim_{n\to\infty}\frac{3}{n}\left\{(9n) - \frac{9}{n^2}\sum_{i=1}^{n}i^2 + \frac{9}{n^2}\sum_{i=1}^{n}i - \frac{9}{4n^2}(n)\right\}$$

$$= \lim_{n\to\infty}\frac{3}{n}\left\{9n - \frac{9}{n^2}\left[\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + \frac{n}{4}\right]\right\}$$

$$= \lim_{n\to\infty}\frac{3}{n}\left\{9n - \frac{9}{n^2}\left[\frac{2n^3+3n^2+n}{6} - \frac{n^2+n}{2} + \frac{n}{4}\right]\right\}$$

$$= \lim_{n\to\infty}\frac{3}{n}\left\{9n - \frac{9}{n^2}\left[\frac{4n^3+6n^2+2n-6n^2-6n+3n}{12}\right]\right\}$$

$$= \lim_{n\to\infty}\frac{3}{n}\left\{9n - \frac{9}{n^2}\left[\frac{4n^3-n}{12}\right]\right\}$$

$$= \lim_{n\to\infty}\frac{3}{n}\left\{9n - 3n + \frac{3}{4n}\right\}$$

$$= \lim_{n\to\infty}\frac{3}{n}\left\{6n + \frac{3}{4n}\right\}$$

$$= \lim_{n\to\infty}\left\{18 + \frac{9}{4n^2}\right\}$$

$$= 18$$

In the next to the last step, the expression inside the braces is the exact area of the n midpoint rectangles.

Upper Riemann Sum

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

$$x_i = x_0 + (i-1)\cdot\Delta x = 0 + (i-1)\cdot\frac{3}{n} = \frac{3i-3}{n}$$

$$\text{Exact Area } = \lim_{n\to\infty}\left\{\sum_{i=1}^{n} f(x_i)\Delta x\right\}$$

$$= \lim_{n\to\infty}\left\{\sum_{i=1}^{n} f\left(\frac{3i-3}{n}\right)\frac{3}{n}\right\}$$

$$= \lim_{n\to\infty}\left\{\sum_{i=1}^{n}\left[9-\left(\frac{3i-3}{n}\right)^2\right]\frac{3}{n}\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{3}{n}\sum_{i=1}^{n}\left[9-\frac{9i^2-18i+9}{n^2}\right]\right\}$$

$$= \lim_{n\to\infty}\frac{3}{n}\left\{\sum_{i=1}^{n}9-\sum_{i=1}^{n}\frac{9i^2}{n^2}+\sum_{i=1}^{n}\frac{18i}{n^2}-\sum_{i=1}^{n}\frac{9}{n^2}\right\}$$

$$= \lim_{n\to\infty}\frac{3}{n}\left\{9n-\frac{9}{n^2}\left[\sum_{i=1}^{n}i^2-\sum_{i=1}^{n}2i+\sum_{i=1}^{n}1\right]\right\}$$

$$= \lim_{n\to\infty}\frac{3}{n}\left\{9n-\frac{9}{n^2}\left[\frac{n(n+1)(2n+1)}{6}-\frac{2n(n+1)}{2}+n\right]\right\}$$

$$= \lim_{n\to\infty}\frac{3}{n}\left\{9n-\frac{9}{n^2}\left[\frac{2n^3+3n^2+n}{6}-\frac{2n^2+2n}{2}+\frac{6n}{6}\right]\right\}$$

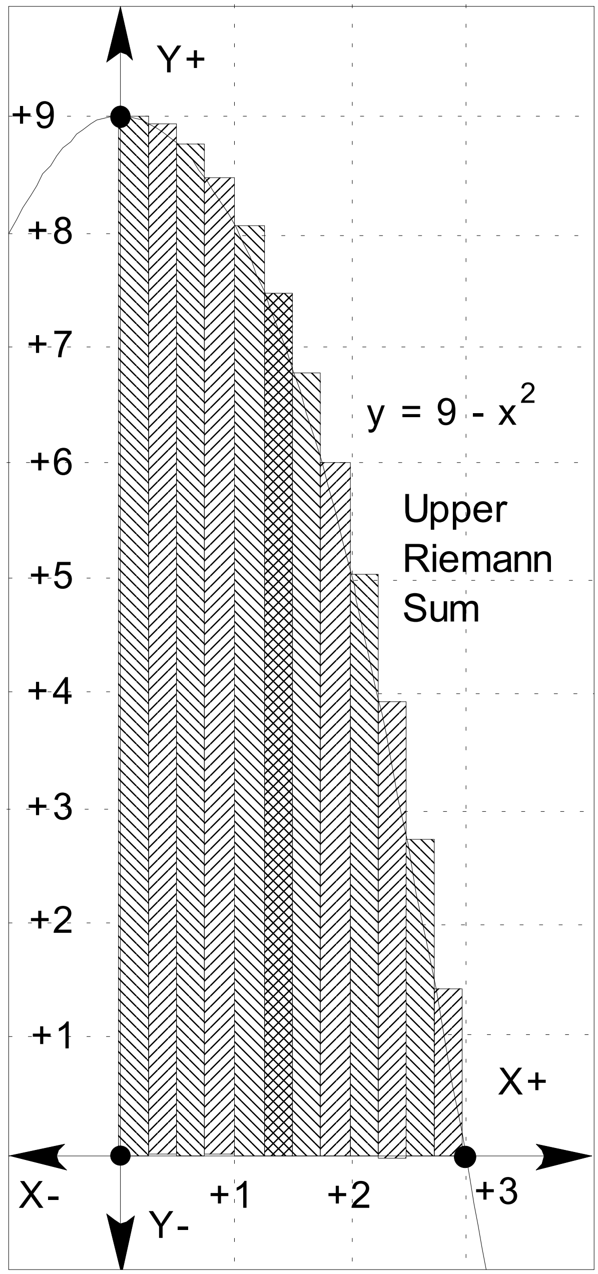
$$= \lim_{n\to\infty}\frac{3}{n}\left\{9n-\frac{9}{n^2}\left[\frac{2n^3+3n^2+n-6n^2-6n+6n}{6}\right]\right\}$$

$$= \lim_{n\to\infty}\frac{3}{n}\left\{9n-\frac{9}{n^2}\left[\frac{2n^3-3n^2+n}{6}\right]\right\}$$

$$= \lim_{n\to\infty}\frac{3}{n}\left\{9n-3n+\frac{9}{2n}-\frac{3}{2n}\right\}$$

$$= \lim_{n\to\infty}\left\{18+\frac{27}{2n}-\frac{9}{2n^2}\right\}$$

$$= 18$$



In the next to the last step, the expression inside the braces is the exact area of the n upper rectangles.

Comparison of the 3 Kinds of Sample Points

	Lower Rectangles	Midpoint Rectangles	Upper Rectangles
n	$18 - \frac{27}{2n} - \frac{9}{2n^2}$	$18 + \frac{9}{4n^2}$	$18 + \frac{27}{2n} - \frac{9}{2n^2}$
4	14.3437500000000000	18.1406250000000000	21.0937500000000000
10	16.6050000000000000	18.0225000000000000	19.3050000000000000
50	17.7282000000000000	18.0009000000000000	18.2682000000000000
100	17.8645500000000000	18.0002250000000000	18.1345500000000000
500	17.9729820000000000	18.0000090000000000	18.0269820000000000
$1,000$	17.9864955000000000	18.0000022500000000	18.0134955000000000
$5,000$	17.9972998200000000	18.0000000900000000	18.0026998200000000
$10,000$	17.9986499550000000	18.0000000225000000	18.0013499550000000
$50,000$	17.9997299982000000	18.0000000009000000	18.0002699982000000