Hints for Graphing Rational Functions

A rational function is any function of the form $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials. To sketch the graph of such a function, perform the following steps in order.

1). First factor $p(x)$ and $q(x)$ but **DO NOT REDUCE** the fraction by canceling common factors. Leave the common factors in both the numerator and denominator. We assume you can find all rational and all irrational real zeros of both $p(x)$ and $q(x)$. We assume the degree of $q(x)$ is at least 1, since otherwise, if $q(x)$ is a constant, $f(x)$ is just a polynomial in disguise.

2). Perform a complete sign analysis on $f(x)$. When you label the critical numbers, any zeros from factors in the denominator make $f(x)$ **undefined**. Only zeros in the numerator $p(x)$ that do not also appear in the denominator should be labeled as 0's.

3). In this step you will determine which of the undefined points found in step 2) are **vertical asymptotes** and which are simply "**holes**" in the graph. All undefined points correspond to either vertical asymptotes or holes. For each $x = a$ value where $q(a) = 0$ we need only consider the multiplicity of the factor $(x - a)$ for both $p(x)$ and $q(x)$. If $p(a) \neq 0$, then the line $x = a$ is a vertical asymptote. In this case, $(x - a)$ is not a factor of $p(x)$. If $p(a) = 0$, then $(x - a)$ is a factor of $p(x)$ and if the multiplicity of this factor is larger than or equal to the multiplicity of this same factor for $q(x)$ then $x = a$ corresponds to a hole. When $(x - a)$ is common factor of both $p(x)$ and $q(x)$ and its multiplicty is greater in $q(x)$, then the line $x = a$ is a vertical asymptote. Further label each undefined point as a hole or a vertical asymptote.

4). If $f(x)$ has any **horizontal asymptote**, it has only one horizontal asymptote. If the degree of $p(x)$ is strictly larger than the degree of $q(x)$ then $f(x)$ does not have any horizontal asymptote. If the degrees of $p(x)$ and $q(x)$ are the same, then the horizontal asymptote is the line $y = k$ where the constant $k = \frac{p_n}{q_n}$ where p_n is the leading coefficient of $p(x)$ and q_n is the leading coefficient of $q(x)$. If the degree of $p(x)$ is less than the degree of $q(x)$ then the line $y = 0$, which is the x-axis, is the horizontal asymptote. In this case, and in this case only, the sign analysis in step 2) will tell you whether the extreme ends of the graph lie either above or below the asymptote. Otherwise, pick a large positive number and a large negative number and compare $f(x)$ with k to determine whether the graph lies above or below the ends of the line that is the horizontal asymptote. You can determine if and where $f(x)$ crosses or touches its horizontal asymptote by solving the equation $f(x) = k$ where $y = k$ is the horizontal asymptote. The equation $f(x) = k$ may have no solutions, or it may have one or more solutions.

5). If the degree of the original $p(x)$ polynomial is larger than the degree of the original $q(x)$ polynomial, then from step 4) we know the graph of $f(x)$ does not have a horizontal asymptote. If the degree of $p(x)$ is exactly one more than the degree of $q(x)$ then the graph of $f(x)$ ***may*** have what is called a **slant asymptote**. Even in this case, a rational function does not have to have a slant asymptote, but like a horizontal asymptote, if $f(x)$ has a slant asymptote, it has only one slant asymptote. The slant asymptote is the line $y = mx + b$, where $mx + b$ is the quotient polynomial that results when you perform long division of polynomials by dividing $q(x)$ into $p(x)$. After performing the long division it helps to write $f(x) = mx + b + \frac{remainder(x)}{q(x)}$ where $remainder(x)$ is another polynomial whose degree is strictly smaller than that of $q(x)$. We implicitly assume the degree of $q(x)$ is at least 1. If $remainder(x)$ is the constant 0, then the graph of $f(x)$ does not have a slant asymptote because $f(x)$ is just a line with holes in it wherever $q(x) = 0$. If $remainder(x)$ is not the constant 0, then the line $y = mx + b$ is a true slant asymptote for $f(x)$. You determine whether the graph is above or below the slant asymptote just like you do for a horizontal asymptote by picking a large positive number and a large negative number for x. The sign of the expression $\frac{remainder(x)}{q(x)}$ for large x determines whether the graph of $f(x)$ is above or below the line $y = mx + b$. The graph of $f(x)$ will touch or cross its slant asymptote line wherever $remainder(x) = 0$ and $q(x) \neq 0$. In particular, if $remainder(x)$ is a polynomial of degree 2 or more, then the graph of $f(x)$ could touch or cross its slant asymptote more than once. It is also possible that simultaneously $remainder(x) = q(x) = 0$ in which case the fraction $\frac{remainder(x)}{q(x)}$ can and should be reduced.

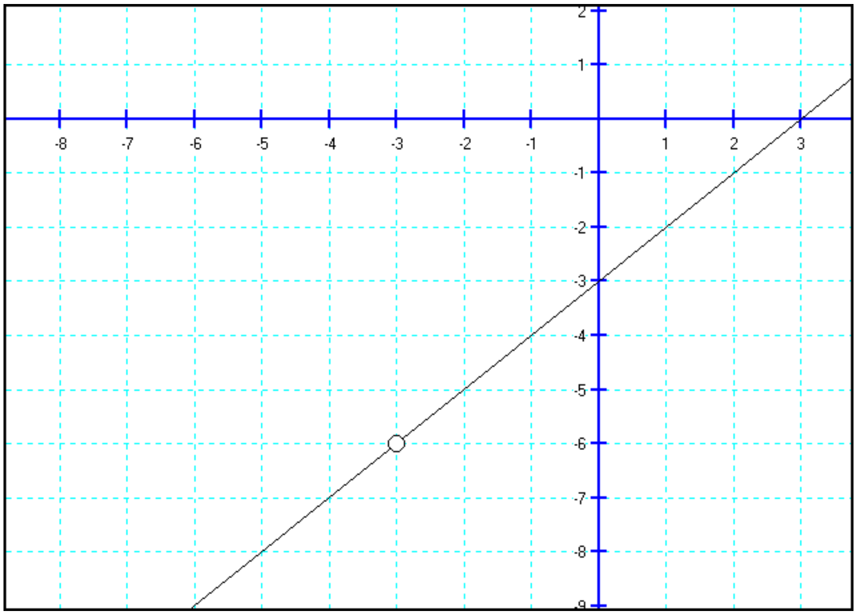
Try graphing the following rational functions:

[1] $f(x) = \frac{x^2 - 9}{x + 3}$ [2] $f(x) = \frac{2}{x^3 - 2x^2 - 3x}$ [3] $f(x) = \frac{x^4 + 3x^3 - 3x^2 - 7x + 6}{x^2 + x - 2}$ [4] $f(x) = \frac{2x^3 + 5x^2 - 20x + 4}{x^2 + 3x - 10}$

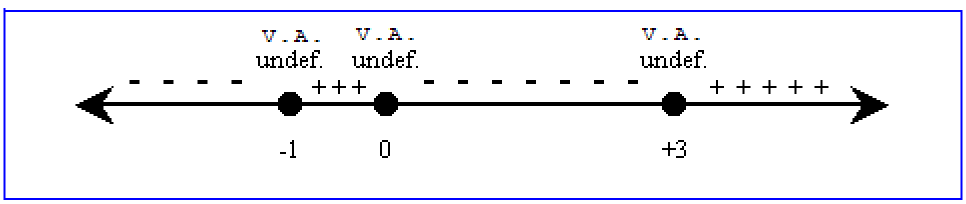
[5] $f(x) = \frac{x^2 - x - 6}{x^2 - 2x}$ [6] $f(x) = \frac{x^3 - 4x^2 + 3x}{x^2 - 4x + 3}$ [7] $f(x) = \frac{x^3 - 3x^2 - 8x + 20}{2x^2 - 2x - 24}$ [8] $f(x) = \frac{x^4 - 1}{x^3 - 1}$

Solutions:

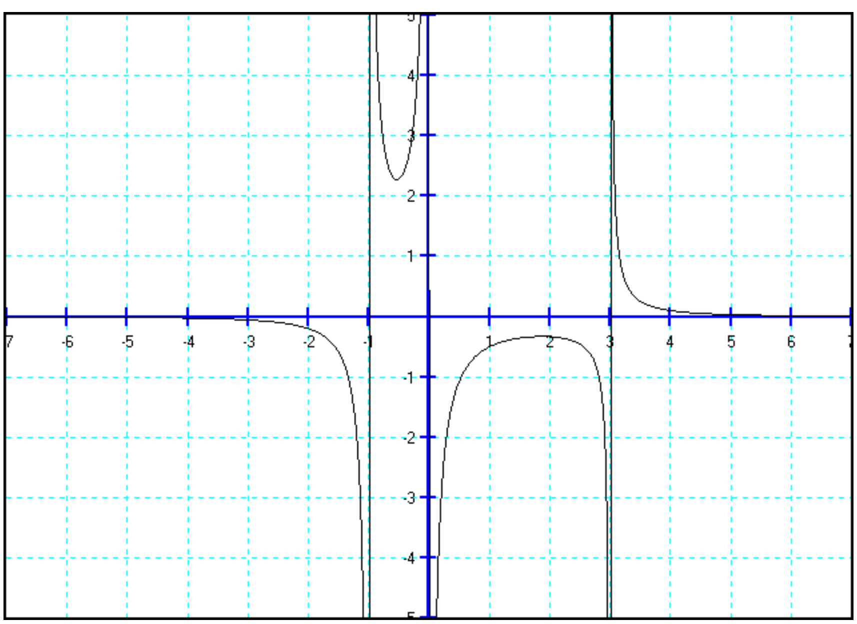
1 $f(x) = \dfrac{x^2 - 9}{x + 3} = \dfrac{(x-3)(x+3)}{(x+3)} = x - 3$ provided $x \neq -3$.
This graph is just a line with a hole in it at $x = -3$.



2 $f(x) = \dfrac{2}{x^3 - 2x^2 - 3x} = \dfrac{2}{x(x^2 - 2x - 3)} = \dfrac{2}{x(x-3)(x+1)}$. The sign analysis for $f(x)$ appears below.



Note that the x-axis is a horizontal asymptote and the sign analysis can be used to tell whether the extremes of the graph are above or below this line. The graph of $f(x)$ appears below. The three vertical asymptotes are $x = -1$; $x = 0$; and $x = 3$.

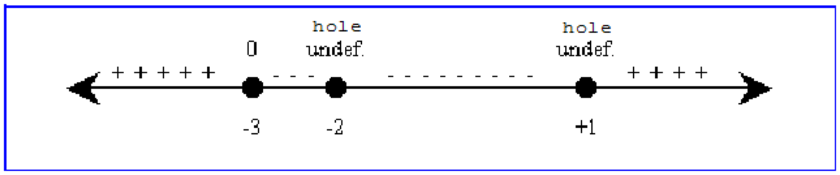


3 $f(x) = \dfrac{x^4 + 3x^3 - 3x^2 - 7x + 6}{x^2 + x - 2}$. We begin by guessing a couple of zeros for the numerator polynomial and applying synthetic substitution. This helps us completely factor the numerator. The denominator is also easy to factor.

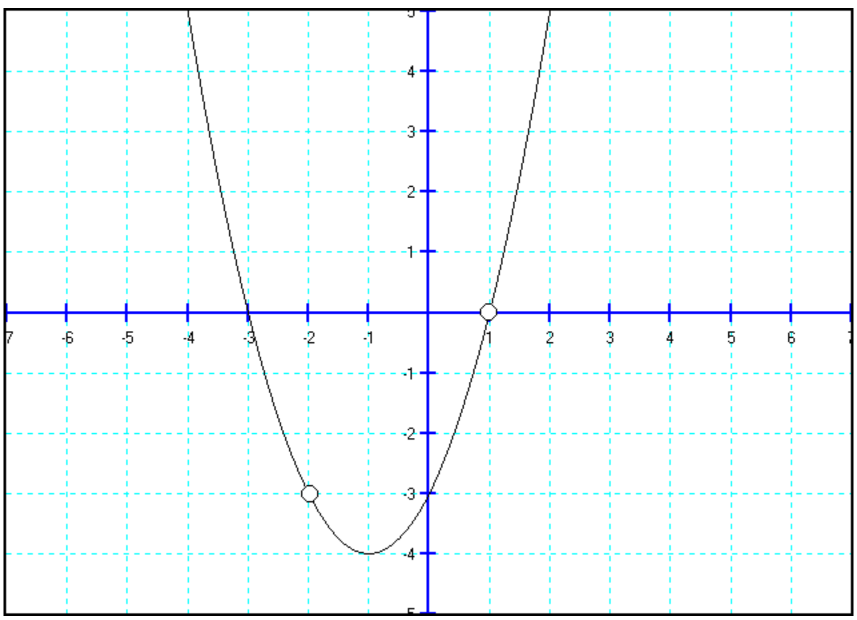
$$\begin{array}{rrrrr|r} 1 & 3 & -3 & -7 & 6 & -2 \\ & -2 & -2 & 10 & -6 & \\ \hline 1 & 1 & -5 & 3 & 0 & \end{array} \qquad \begin{array}{rrrr|r} 1 & 1 & -5 & 3 & 1 \\ & 1 & 2 & -3 & \\ \hline 1 & 2 & -3 & 0 & \end{array}$$

$$f(x) = \frac{(x+2)(x^3 + x^2 - 5x + 3)}{(x+2)(x-1)} = \frac{(x+2)(x-1)(x^2 + 2x - 3)}{(x+2)(x-1)} = \frac{(x+2)(x-1)(x+3)(x-1)}{(x+2)(x-1)}$$

$= (x+3)(x-1)$ provided $x \neq -2$ and $x \neq 1$. The sign analysis for $f(x)$ appears below.



Note that the graph of $f(x)$ is just a parabola with two holes in it.



4 $f(x) = \dfrac{2x^3 + 5x^2 - 20x + 4}{x^2 + 3x - 10}$. We can guess one root to the numerator polynomial by peforming synthetic substitution.

$$\begin{array}{rrrr|r} 2 & 5 & -20 & 4 & 2 \\ & 4 & 18 & -4 & \\ \hline 2 & 9 & -2 & 0 & \end{array}$$

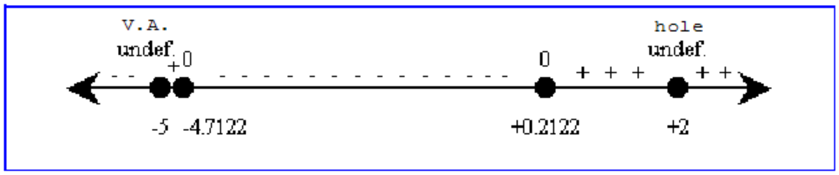
$$f(x) = \frac{(x-2)(2x^2 + 9x - 2)}{(x+5)(x-2)}$$

To completely factor the numerator we must solve the quadratic equation: $2x^2 + 9x - 2 = 0$.

When we do this, we find the two roots are $x = \dfrac{-9 \pm \sqrt{97}}{4}$. We will call the two roots a and b where $a = \dfrac{-9 - \sqrt{97}}{4} \approx -4.7122$ and $b = \dfrac{-9 + \sqrt{97}}{4} \approx 0.2122$. Then a and b are critical numbers and we may write $f(x)$ in its completely factored form as:

$$f(x) = \frac{(x-2)(x-a)(x-b)}{(x+5)(x-2)}$$

The sign analysis for $f(x)$ is:



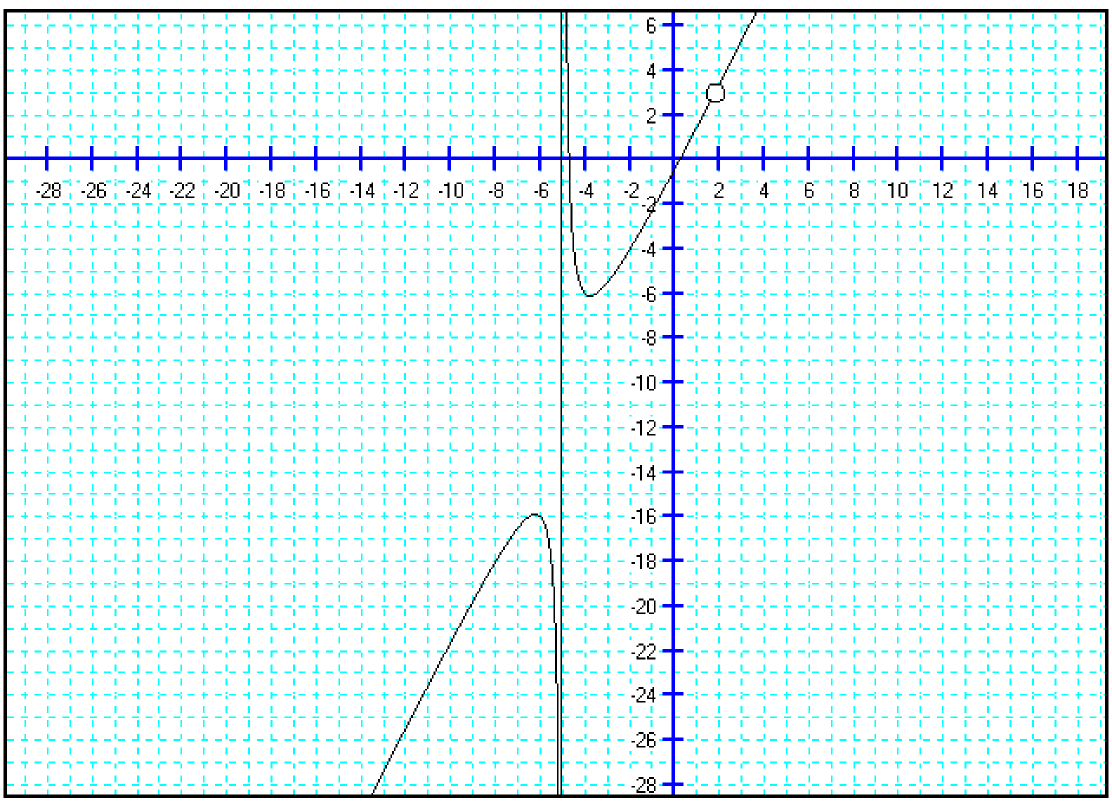
Because the points -5 and $a = -4.7122$ are so close together it may be difficult to read the space between these points, but the sign over the short interval between -5 and -4.7122 is +.

We note that the line $x = -5$ is the only vertical asymptote for $f(x)$. $f(x)$ is undefined and has a hole at $x = 2$. Since the degree of the numerator is exactly one more than the degree of the denominator we look for a slant asymptote by performing long division.

$$\begin{array}{r|rrrrrrr} & 2x & - & 1 & & & & \\ \hline x^2 + 3x - 10 & 2x^3 & + & 5x^2 & - & 20x & + & 4 \\ & 2x^3 & + & 6x^2 & - & 20x & & \\ \hline & & & -x^2 & + & 0x & + & 4 \\ & & & -x^2 & - & 3x & + & 10 \\ \hline & & & & & 3x & - & 6 \end{array}$$

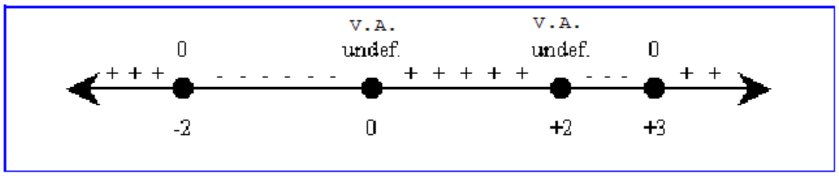
$$f(x) = (2x-1) + \frac{3x-6}{x^2+3x-10} = (2x-1) + \frac{3(x-2)}{(x+5)(x-2)} = (2x-1) + \frac{3}{(x+5)} \text{ provided } x \neq 2$$

This shows the line $y = 2x - 1$ is a slant asymptote. The graph of $f(x)$ appears below. The slant asymptote is not drawn.



$\boxed{5}$ $f(x) = \dfrac{x^2 - x - 6}{x^2 - 2x} = \dfrac{(x-3)(x+2)}{x(x-2)}$

The sign analysis for $f(x)$ is:

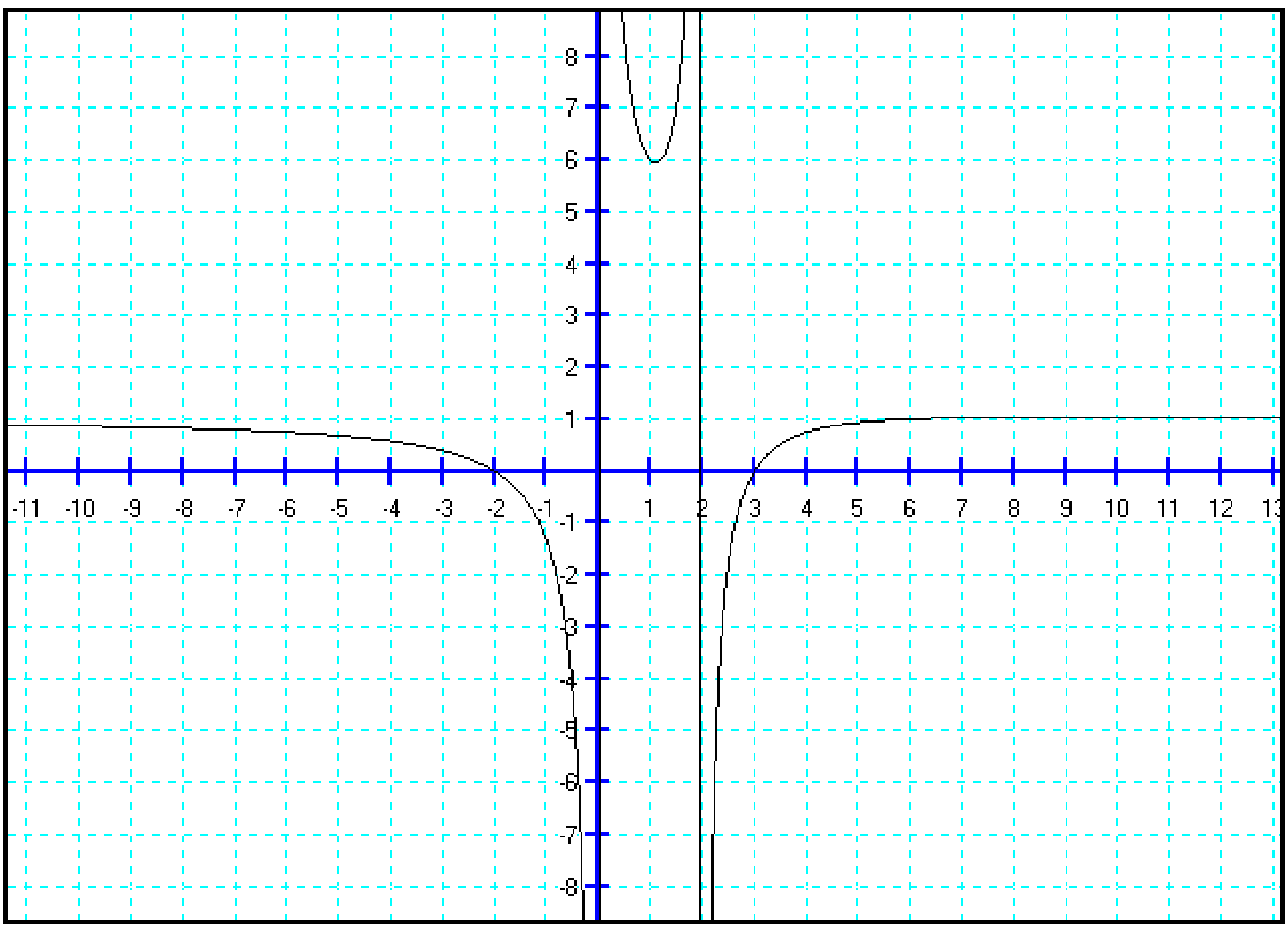


Since the degrees of both the numerator and denominator are the same, the line $y = \frac{1}{1} = 1$ is a horizontal asymptote.

To determine whether the graph is above or below the horizontal asymptote on the far left, we let $x = -100$ and compute $y = \dfrac{(-103)\cdot(-98)}{(-100)(-102)} = \dfrac{10094}{10200} \approx 0.989607 < 1$. On the left, the graph is below the line $y = 1$.

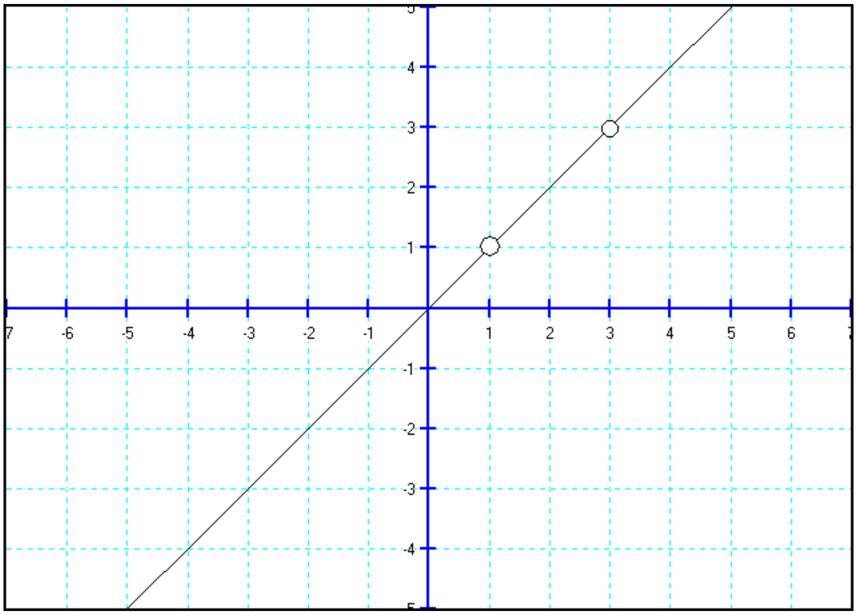
To determine whether the graph is above or below the horizontal asymptote on the far right, we let $x = +100$ and compute $y = \dfrac{(97)\cdot(102)}{(100)(98)} = \dfrac{9894}{9800} \approx 1.00959 > 1$. On the right, the graph is above the line $y = 1$.

The graph of $f(x)$ appears below. The lines $x = 0$ and $x = 2$ are both vertical asymptotes.



6 $f(x) = \dfrac{x^3 - 4x^2 + 3x}{x^2 - 4x + 3} = \dfrac{x(x^2 - 4x + 3)}{(x-3)(x-1)} = \dfrac{x(x-3)(x-1)}{(x-3)(x-1)} = x$ provided $x \neq 1$ and $x \neq 3$.

This graph is just a line with two holes in it. Note that even though the degree of the numerator is exactly one more than the degree of the denominator, there is no slant asymptote. The graph of $f(x)$ appears below.



7 $f(x) = \dfrac{x^3 - 3x^2 - 8x + 20}{2x^2 - 2x - 24}$

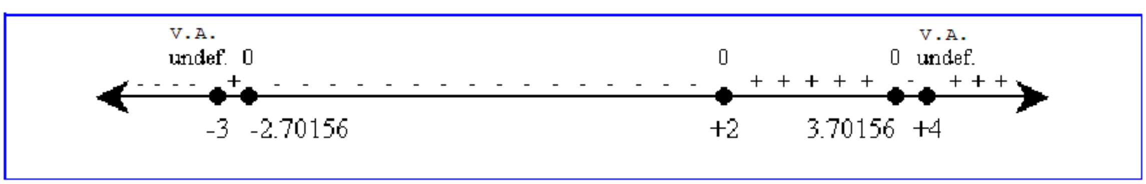
We begin by trying to factor the numerator. We guess $x = 2$ is a root and try synthetic substitution.

$$\begin{array}{rrrr|l} & & & & 2 \\ 1 & -3 & -8 & 20 & \\ & 2 & -2 & -20 & \\ \hline 1 & -1 & -10 & 0 & \end{array}$$

So we may now write $f(x) = \dfrac{(x-2)(x^2 - x - 10)}{2(x^2 - x - 12)} = \dfrac{(x-2)(x^2 - x - 10)}{2(x-4)(x+3)}$

The discriminant of the quadratic $x^2 - x - 10 = 0$ is 41 so the remaining two roots of the numerator are irrational. We let $a = \dfrac{1 - \sqrt{41}}{2} \approx -2.70156$ and $b = \dfrac{1 + \sqrt{41}}{2} \approx 3.70156$ denote the two roots to this quadratic.

Then $f(x) = \dfrac{(x-2)(x-a)(x-b)}{2(x-4)(x+3)}$ and the sign analysis for $f(x)$ is:



The two undefined points correspond to vertical asymptotes, $x = -3$ and $x = 4$.

Since the degree of the numerator polynomial is exactly one more than that of the denominator, we perform long division to see if there is a slant asymptote for this graph.

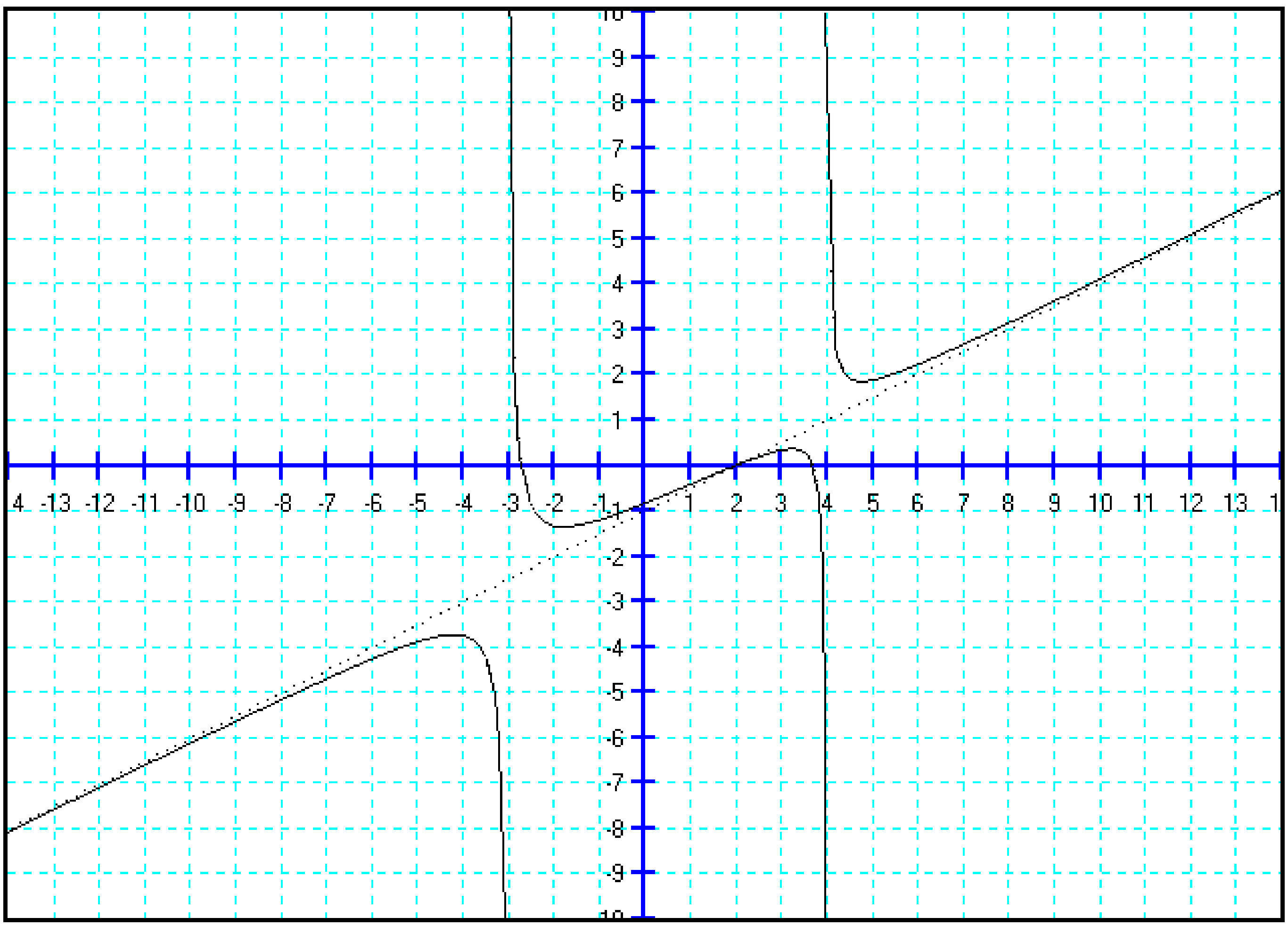
$$
\begin{array}{r|rrrrrrr}
 & \frac{1}{2}x & - & 1 & & & & \\
\hline
2x^2 - 2x - 24 & x^3 & - & 3x^2 & - & 8x & + & 20 \\
 & x^3 & - & x^2 & - & 12x & & \\
\hline
 & & & -2x^2 & + & 4x & + & 20 \\
 & & & -2x^2 & + & 2x & + & 24 \\
\hline
 & & & & & 2x & - & 4
\end{array}
$$

Since the remainder is nonzero we have a slant asymptote and we may write

$$f(x) = (\tfrac{1}{2}x - 1) + \frac{2x-4}{2(x-4)(x+3)} = (\tfrac{1}{2}x - 1) + \frac{2(x-2)}{2(x-4)(x+3)} = (\tfrac{1}{2}x - 1) + \frac{x-2}{(x-4)(x+3)}$$

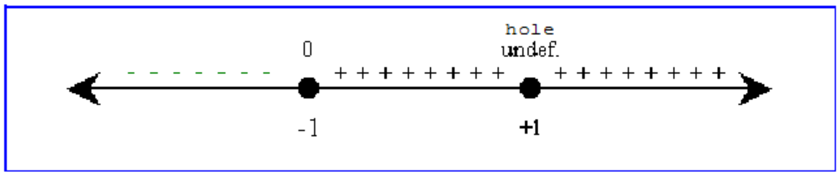
The line $y = \frac{1}{2}x - 1$ is the slant asymptote.

The graph of $f(x)$ appears below. This particular function has the unusual property that it crosses its slant asymptote at the point $(2,0)$ and this point is also an x-intercept in the graph. The graph lies below the slant asymptote when $x < -3$ and it lies above the slant asymptote when $x > 4$.



8 $f(x) = \frac{x^4-1}{x^3-1} = \frac{(x^2-1)(x^2+1)}{(x-1)(x^2+x+1)} = \frac{(x-1)(x+1)(x^2+1)}{(x-1)(x^2+x+1)} = \frac{(x+1)(x^2+1)}{x^2+x+1}$ provided $x \neq 1$.

The sign analysis for $f(x)$ is:



We can see that at $x = 1$ we have a hole in the graph.

Since the degree of the numerator is exactly one more than that of the denominator we perform long division to write $f(x)$ in the form:

$$f(x) = x + \frac{x-1}{x^3-1} = x + \frac{(x-1)\cdot 1}{(x-1)\cdot(x^2+x+1)} = x + \frac{1}{x^2+x+1} \text{ provided } x \neq 1.$$

From this last equation we can determine that $f(x)$ has a slant asymptote that is the line $y = x$.
The graph of $f(x)$ is shown below. It is interesting to note that $f(x)$ lies all on one side (above) its slant asymptote.