The Number e and Compound Interest

by

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some deliberate reason, this page is intentionally blank!

e The Number of the Natural Logarithm Base

The mathematical number $\boldsymbol{e} \approx 2.71828182846$ can be understood to be a special number which occurs in mathematics, somewhat like $\boldsymbol{\pi} \approx 3.14159265359$. Although $\boldsymbol{\pi}$ occurs naturally in the geometry of the circle, it is surprising that $\boldsymbol{e}$ occurs in nature as part of compound growth. Both $\boldsymbol{\pi}$ and $\boldsymbol{e}$ are irrational transcendental numbers. Being irrational means they cannot be written as exact fractions, or another way of saying the same thing, they cannot be represented by repeating decimals. Being transcendental means they are also not roots of any polynomial equation which has rational coefficients.

In the following we give the financial formula for the future value associated with the compound interest earnings of a fixed amount and also the formula for the future value of a series of periodic payments. The following five variables are the standard variables used in the math of finance.

FV = the future value amount
PV = the present value amount
i = the periodic interest rate applied over one time period
n = the number of time periods
PMT = the amount of the periodic payment

There are only two financial formulas which are required to solve most problems involving compound interest of either a single fixed payment or a series of periodic payments. The future value FV is the same in both formulas and can be used to equate a loan amount with the amount of each periodic payment for that loan. Only the first formula is needed to introduce the famous number $\boldsymbol{e}$.

$$\boldsymbol{FV = PV \cdot (1 + i)^n} \quad \textbf{and} \quad \boldsymbol{FV = PMT \cdot \left[\frac{(1 + i)^n - 1}{i}\right]}$$

A TYPICAL CAR LOAN

For example, suppose you come to me to get a 3-year \$8,000 car loan. Assuming an annual interest rate of 12% per year, which is equivalent to 1% per month, I can use the first formula to decide the future value of my \$8,000 thirty-six months from now. I figure this amount before I decide to loan you the money! In fact, I need to know this amount to be able to determine what your monthly car payments to me will be. Substituting, PV=8000, $i = 0.01$ and $n = 36$ we use the first formula to calculate the future value FV.

$$\begin{aligned} FV &= PV \cdot (1 + i)^n \\ &= \$8000 \cdot (1 + 0.01)^{36} \\ &= \$8000 \cdot (1.01)^{36} \\ &= \$8000 \cdot [1.43076878359] \\ &= \$11{,}446.15 && \{ \text{ rounded to the nearest penny } \} \end{aligned}$$

So I expect my money "multiply" by the factor in the square brackets which is about 1.43 which is not quite $1\frac{1}{2}$ times. The quantity $(1+i)^n$ is significant because it always acts as the multiplier of your money (or mine!). We will next show how this quantity also determines the number $\boldsymbol{e}$!

Although car loan problems are usually figured on a monthly basis, it will be interesting to consider the value of money when it is compounded more often that once every month. Note that the values of n and i must constantly be adjusted to be consistent with the stated interest rate. In fact, there is always a time period associated with an interest rate.

If your bank tells you your money is earning 8% interest they are not telling you the whole truth. The bank's quoted 8% interest rate is probably a nominal yearly rate which would correspond to $8\% \div 12 = \frac{2}{3}\%$ or 0.666% per month. Similarly, when your credit card company tells you they are only charging you $1\frac{1}{2}\%$ interest, that is a monthly rate which translates to $1.5\% \times 12 = 18\%$ per year.

To keep things simple, let us just consider the future value of $1 which is compounded for the time periods listed in the table below. You can choose any interest rate per year, but to keep things simple we will use 100% per year. We don't mean to be greedy, but please note that this is only 8.3% per month, or 0.27397% per day, or 0.01141% per hour, or 0.000190% per minute!

Time Period Length	Time Periods Per Year $= n$	Interest Rate Per Time Period $= i$	Compounding Factor $= (1+i)^n$
Year	1	1.0	2.0
Month	12	$\frac{1}{12} = 0.0833333333$	2.61303529013
Day	365	$\frac{1}{365} = 0.002739726027$	2.71456748203
Hour	$365 \cdot 24 = 8760$	$\frac{1}{8760} = 0.000114155251142$	2.71812669163
Minute	$8760 \cdot 60 = 525600$	$\frac{1}{525600}$0.00000190258751903	2.71827924258
Second	$525600 \cdot 60 = 31536000$	$\frac{1}{31536000} = 0.0000000317097919838$	2.71828178536

To compound every instant, take $\lim_{n \to \infty} (1+\frac{1}{n})^n = \boldsymbol{e} \approx 2.71828182846$

WARNING! If you attempt the above calculations on an ordinary calculator using a $\boxed{y^x}$ key you may get very inaccurate results. Most calculators are not designed to compute $(1+i)^n$ when i is very small and n is very large. Special precautions must be used to insure round-off errors don't interfere with the calculation.

Given the above expression for $\boldsymbol{e}$, we can prove:

$$\lim_{n \to \infty} (1 + \tfrac{x}{n})^n = \boldsymbol{e}^x$$

Proof:

$$\boldsymbol{e} = \lim_{n \to \infty} (1 + \tfrac{1}{n})^n$$ (This is a definition of $\boldsymbol{e}$.)

$$\boldsymbol{e}^x = \left\{ \lim_{n \to \infty} (1 + \tfrac{1}{n})^n \right\}^x$$ (Assume x is a fixed real number. Take the xth power on both sides of the above equation.)

$$= \lim_{n \to \infty} \left\{ (1 + \tfrac{1}{n})^n \right\}^x$$ (Justification for moving the xth power inside the limit depends on continuity of the xth power function. The xth power of the limit is the limit of the xth power.)

$$= \lim_{n \to \infty} (1 + \tfrac{1}{n})^{nx}$$ (Apply the multiplication rule for exponents since we are taking a power of a power.)

$$= \lim_{n \to \infty} (1 + \tfrac{x}{nx})^{nx}$$ (Multiply $\frac{1}{n}$ by $\frac{x}{x} = 1$)

$$= \lim_{nx \to \infty} (1 + \tfrac{x}{nx})^{nx}$$ (As $n \to \infty$ so does nx and vice versa!)

$$= \lim_{k \to \infty} (1 + \tfrac{x}{k})^k$$ (Replace nx with the single variable k)

$$= \lim_{n \to \infty} (1 + \tfrac{x}{n})^n$$ (n is just as good a variable as k!)

CONTINUOUS COMPOUND INTEREST

The previous two standard financial formulas can also be used to determine formulas for continuous compounding. As is shown below, the same formulas hold with the factor $(1+i)^n$ replaced by $\boldsymbol{e}^{in}$.

A Lump Sum Compounded Continuously For n Time Periods

$$FV = PV \cdot (1 + i)^n \qquad \{ \text{discrete compounding} \}$$

$$= PV \cdot (1 + \tfrac{i}{m})^{n \cdot m} \qquad \{ \text{compounded } n \cdot m \text{ times at a rate } = \tfrac{i}{m} \}$$

$$= PV \cdot [(1 + \tfrac{i}{m})^m]^n$$

To arrive at continuous compounding we simply take the limit as m approaches ∞.

$$= PV \cdot [\boldsymbol{e}^i]^n \qquad \{ \boldsymbol{e}^i = \text{the limit as } m \to \infty \text{ of } (1 + \tfrac{i}{m})^m \}$$

$$= PV \cdot \boldsymbol{e}^{i \cdot n}$$

This example shows that when compounding continuously, $(1+i)^n$ can be replaced by $\boldsymbol{e}^{i \cdot n}$, or, equivalently, i can be replaced by $\boldsymbol{e}^i - 1$.

$$(1+i)^n \approx \boldsymbol{e}^{i \cdot n} \qquad\qquad i \approx \boldsymbol{e}^i - 1$$

Note that in the above and in what follows i is the periodic interest rate which can be described as the interest rate per time period, and n denotes the number of time periods.

Series Of N Equal Payments Compounded Continuously Over n Time Periods

$$FV = PMT \cdot (1+i)^{n-1} + PMT \cdot (1+i)^{n-2} + \cdots + PMT \cdot (1+i) + PMT$$

$$\approx PMT \cdot [e^{i \cdot (n-1)}] + PMT \cdot [e^{i \cdot (n-2)}] + \cdots + PMT \cdot [e^{i \cdot 1}] + PMT$$

$$= PMT \cdot [(e^i)^{n-1} + (e^i)^{n-2} + \cdots + (e^i) + 1]$$

$$= PMT \cdot [1 + (e^i) + (e^i)^2 + (e^i)^3 + \cdots + (e^i)^{n-1}]$$

$$= PMT \cdot [\tfrac{1 - (e^i)^n}{1 - (e^i)}]$$

$$= PMT \cdot [\tfrac{e^{i \cdot n} - 1}{e^i - 1}]$$

$$= PMT \cdot [\tfrac{e^{i \cdot n} - 1}{i}]$$

$$= \frac{PMT}{i} \cdot [\, e^{\, i \cdot n} \; - \; 1 \,]$$ { this is the standard form }

{ for those who know calculus this standard form can be expanded as shown below to arrive at an equivalent integral form }

$$= \; PMT \cdot \frac{1}{i} \cdot [\, e^{\, i \cdot n} \; - \; e^{\,0} \,]$$

$$= \; PMT \cdot \frac{1}{i} \cdot [\, e^{\, i \cdot t} \,|\, _{0}^{n} \,]$$

$$= \; PMT \cdot [\, \frac{e^{\, i \cdot t}}{i} \,|\, _{0}^{n} \,]$$

$$= \; PMT \cdot \int_{0}^{n} e^{\, i \cdot t} \; dt$$ { this is the integral form }

Unequal Payment And Compounding Periods

For those interested in more mathematical details, below we provide formulas which can be used to calculate annuities where the payment and compounding frequencies are different.

Sometimes a series of payments are made where the payment period is more frequent than the compounding period. Suppose a series of $n \cdot m$ equal payments will be made over n time periods in which there will be m payments per time period accumulating an interest rate i per time period. Then the discrete and continuous future value formulas are as follows:

$$FV \; = \; \frac{m}{i} \cdot PMT \cdot [\, (\, 1 \; + \; \frac{i}{m} \,)^{\, n \cdot m} \; - \; 1 \,]$$ { discrete case }

$$FV \; = \; \frac{m}{i} \cdot PMT \cdot [\, e^{\, i \cdot n} \; - \; 1 \,]$$ { continuous case }

$$FV \; = \; m \cdot PMT \cdot \int_{0}^{n} e^{\, i \cdot t} \; dt$$

{ this is the integral form of the continuous case }

Previously we applied a limit process to a discrete variable n which takes on counting values to establish relationships with the number ***e***. An alternate proof of the limit approximation for ***e*** can be applied which employs L'Hospital's Rule. But this derivation requires a prior knowledge of some calculus.

Let $y = (1 + \frac{1}{x})^x$

Then, $ln(y) = x \cdot ln(1 + \frac{1}{x})$ Take logs on both sides.

$$\lim_{x \to \infty} [\, ln(y) \,] = \lim_{x \to \infty} \left[\, x \cdot ln(1 + \tfrac{1}{x}) \,\right]$$ Apply $\lim_{x \to \infty}$ to both sides.

$$= \lim_{x \to \infty} \left[\frac{ln(1 + \frac{1}{x})}{\frac{1}{x}} \right]$$ Dividing by $\frac{1}{x}$ is the same as multiplying by x.

$$= \lim_{x \to \infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot D_x\left(\frac{1}{x}\right)}{D_x\left(\frac{1}{x}\right)}$$ Apply L'Hospital's Rule.

$$= \lim_{x \to \infty} \frac{1}{1 + \frac{1}{x}}$$ Cancel $D_x\left(\frac{1}{x}\right)$ from both numerator and denominator.

$$= 1$$ $\lim_{x \to \infty} \frac{1}{1 + \frac{1}{x}} = 1$

$$e^{\lim_{x \to \infty} [\, ln(y) \,]} = e^1$$ Exponentiate on both sides.

$$\lim_{x \to \infty} e^{[\, ln(y) \,]} = e$$ Take e inside the limit.

$$\lim_{x \to \infty} \left[e^{ln(y)} \right] = e$$ Simplify.

$$\lim_{x \to \infty} [\, y \,] = e$$ $y = e^{ln(y)}$

$$\lim_{x \to \infty} (1 + \tfrac{1}{x})^x = e$$ Substitute for y.