A Limit Proof Technique

One of the most difficult tasks students face in beginning calculus is writing formal proofs for limits. The concluding line in all such proofs is that $|f(x) - L| < \epsilon$ where the original limit problem is to show that $\lim_{x \to a} f(x) = L$.

In many cases, but not in all cases, we can sometimes factor $f(x) - L$ as a product in the form $(x-a) \cdot g(x)$ where $g(x)$ is just another function comprising the second factor. Note the first factor must be of the form $(x-a)$ where the limit is being taken as x approaches a. When such a factorization exists, then all that needs to be shown is that the function $g(x)$ is bounded over some open interval containing a.

When $g(x)$ is bounded for x near a we can guarantee that $|f(x) - L| < \epsilon$ by making the equivalent product $(x-a) \cdot g(x)$ sufficiently small. If $|g(x)| < B,$ we can make the product $(x-a) \cdot g(x)$ small because we can make the first factor $(x-a)$ very small, even when $g(x)$ is as large as B, by insuring x is sufficiently close to a.

More formally, let's suppose for some positive constant B that we can show that for all x on both sides of a, but near a, that $|g(x)| < B$. When we say for x near a we just mean x is within some neighborhood of a, say x is within c unit$s$ of a. $a - c < x < a + c$. i.e. $|x - a| < c$. For many functions we can take $c = 1$, but otherwise c need only be some small positive number. Then we can construct a formal proof for the limit as follows.

Let $\epsilon > 0$ be given. Choose $\delta = minimum\left\{c, \dfrac{\epsilon}{B}\right\}$. Assume $0 < |x - a| < \delta$. Then we can write $|x - a| < c$ and with x restricted by this last inequality we should be able to derive that $|g(x)| < B$.

Furthermore, we can write $|x - a| < \dfrac{\epsilon}{B}$. Now since $0 < |x - a| < \dfrac{\epsilon}{B}$ and since $0 \le |g(x)| < B$, we can multiply the middle and right parts of these two inequalities and compare the smaller product with the larger product.

$$|x - a| < \frac{\epsilon}{B}$$

$$|x - a| \cdot |g(x)| < \frac{\epsilon}{B} \cdot |g(x)| < \frac{\epsilon}{B} \cdot B = \epsilon$$

$$|(x - a) \cdot g(x)| < \epsilon$$

$$|f(x) - L| < \epsilon \qquad Q.E.D.$$

$(nota\, bene)$

In actual practice, you will want to find the $g(x)$ function first, and then try to establish the upper bound B for that function. Establishing the value of B may naturally lead to other restictions on either x or a or both that may also influence the cases in your proof and may influence your choice of δ.

Example 1: Prove that $\lim_{x \to a} x^4 = a^4$ where a may be positive or zero or negative.

Let $\epsilon > 0$ be given. Choose $\delta = minimum \left\{ 1, \frac{\epsilon}{4(|a|+1)^3} \right\}$.

Assume $0 < |x - a| < \delta$.

Then $|x - a| < 1$ so that $-1 < x - a < 1$, or $a - 1 < x < a + 1$.
Since $|a - 1| \leq |a| + 1$ and since $|a + 1| \leq |a| + 1$ we know
$|a| + 1 \geq maximum\{|a - 1|, |a + 1|\}$. Thus we know $|x| < |a| + 1$.
We also know that $|a| < |a| + 1$.

Finally, $x^4 - a^4 = (x - a) \cdot \{x^3 + ax^2 + a^2x + a^3\}$ and so we can derive an upper bound for the second factor by writing:

$|x^3 + ax^2 + a^2x + a^3| \leq |x^3| + |ax^2| + |a^2x| + |a^3| =$
$|x|^3 + |a||x|^2 + |a|^2|x| + |a|^3 < (|a|+1)^3 + (|a|+1)^3 + (|a|+1)^3 + (|a|+1)^3 =$
$4(|a|+1)^3$.

We have just shown that $|x^3 + ax^2 + a^2x + a^3| < 4(|a|+1)^3$.

We also have that $|x - a| < \frac{\epsilon}{4(|a|+1)^3}$.

Now multiplying these last two inequalities we can write

$$|x - a| \cdot |x^3 + ax^2 + a^2x + a^3| < \frac{\epsilon}{4(|a|+1)^3} \cdot 4(|a|+1)^3$$

$$|x - a| \cdot |x^3 + ax^2 + a^2x + a^3| < \epsilon$$

$$|(x - a) \cdot \{x^3 + ax^2 + a^2x + a^3\}| < \epsilon$$

$$\left|x^4 - a^4\right| < \epsilon$$

Q.E.D.

Example 2: Prove that $\lim_{x \to a} \sqrt[3]{x} = \sqrt[3]{a}$ where $a \neq 0$.

Let $\epsilon > 0$ be given. Choose $\delta = minimum \left\{ \frac{|a|}{2}, \epsilon\left(\sqrt[3]{a^2}\right) \right\}$. Assume $0 < |x-a| < \delta$.

Then $|x-a| < \frac{|a|}{2}$ so that $-\frac{|a|}{2} < x - a < \frac{|a|}{2}$, or $a - \frac{|a|}{2} < x < a + \frac{|a|}{2}$.
Now regardless of the sign of a, we claim that x and a have the same sign.
For if $a < 0$ then the inequality that $x < a + \frac{|a|}{2}$ means $x < a - \frac{a}{2} = \frac{a}{2} < 0$.
If $a > 0$ then the inequality that $a - \frac{|a|}{2} < x$ means $0 < \frac{a}{2} = a - \frac{a}{2} < x$.
Since x and a have the same sign we know $\sqrt[3]{xa} > 0$.

Now we can write that $0 < \left| \frac{1}{\left\{ \left(\sqrt[3]{x}\right)^2 + \left(\sqrt[3]{x}\right)\left(\sqrt[3]{a}\right) + \left(\sqrt[3]{a}\right)^2 \right\}} \right| =$

$$= \frac{1}{\left| \left\{ \left(\sqrt[3]{x}\right)^2 + \sqrt[3]{xa} + \left(\sqrt[3]{a}\right)^2 \right\} \right|} = \frac{1}{\left(\sqrt[3]{x}\right)^2 + \sqrt[3]{xa} + \left(\sqrt[3]{a}\right)^2} < \frac{1}{\left(\sqrt[3]{a}\right)^2}$$

We dropped the first two positive denominator values in order to establish this last inequality.

Now multiplying the middle and right parts of the two inequalities that

$$0 < |x-a| < \epsilon\left(\sqrt[3]{a^2}\right) \text{ and } 0 < \left| \frac{1}{\left\{ \left(\sqrt[3]{x}\right)^2 + \left(\sqrt[3]{x}\right)\left(\sqrt[3]{a}\right) + \left(\sqrt[3]{a}\right)^2 \right\}} \right| < \frac{1}{\left(\sqrt[3]{a}\right)^2}$$

we have $|x-a| \cdot \left| \frac{1}{\left\{ \left(\sqrt[3]{x}\right)^2 + \left(\sqrt[3]{x}\right)\left(\sqrt[3]{a}\right) + \left(\sqrt[3]{a}\right)^2 \right\}} \right| < \epsilon\left(\sqrt[3]{a^2}\right) \cdot \frac{1}{\left(\sqrt[3]{a}\right)^2}$

$$|x-a| \cdot \left| \frac{1}{\left\{ \left(\sqrt[3]{x}\right)^2 + \left(\sqrt[3]{x}\right)\left(\sqrt[3]{a}\right) + \left(\sqrt[3]{a}\right)^2 \right\}} \right| < \epsilon$$

$$|x-a| \cdot \left| \frac{\left\{ \sqrt[3]{x} - \sqrt[3]{a} \right\} \cdot 1}{\left\{ \sqrt[3]{x} - \sqrt[3]{a} \right\} \cdot \left\{ \left(\sqrt[3]{x}\right)^2 + \left(\sqrt[3]{x}\right)\left(\sqrt[3]{a}\right) + \left(\sqrt[3]{a}\right)^2 \right\}} \right| < \epsilon$$

$$|x-a| \cdot \frac{\left| \sqrt[3]{x} - \sqrt[3]{a} \right|}{|x-a|} < \epsilon$$

$$\left| \sqrt[3]{x} - \sqrt[3]{a} \right| < \epsilon$$

$Q.E.D.$

Example 3: Prove that $\lim\limits_{x \to 0} \sqrt[3]{x} = 0$.

If we try the factoring technique we would write $x \cdot \dfrac{\sqrt[3]{x}}{x}$ for $(x-a) \cdot (f(x) - L)$.

This implies the $g(x)$ function is $\dfrac{\sqrt[3]{x}}{x} = \dfrac{1}{x^{\frac{2}{3}}} = \dfrac{1}{\sqrt[3]{x^2}}$.

Unfortunately, the function $\dfrac{1}{\sqrt[3]{x^2}}$ is not bounded within any neighborhood of 0.

This is an example where the technique explained before Example 1 cannot be applied.

However, both Example 2 and Example 3 can be solved more easily and more directly by giving a more geometric argument based on the graph of the function $y = \sqrt[3]{x}$.

We can prove the limit in Example 3 by letting $\epsilon > 0$ be given and choosing $\delta = \sqrt[3]{\epsilon}$.
Then assume $0 < |x| < \delta$.

Then $|x| < \sqrt[3]{\epsilon}$. This implies that $-\sqrt[3]{\epsilon} < x < \sqrt[3]{\epsilon}$.

Because the function $h(x) = x^3$ is monotonically increasing we can apply this function to all three parts of this last inequality to establish that

$\left(-\sqrt[3]{\epsilon}\right)^3 < x^3 < \left(\sqrt[3]{\epsilon}\right)^3$. i.e., $-\epsilon < x^3 < \epsilon$.

This means $|x^3 - 0| < \epsilon$ which is what we needed to establish.

An easier proof for Example 2 can also be given. However, looking back at Example 2 we can understand that the expression $\dfrac{|a|}{2}$ was used as a possible first choice for δ just to insure that x and a have the same sign. When $a \neq 0$, and x is very close to a then x and a have the same sign and this property insures that the middle term in the denominator, namely $\sqrt[3]{x}\,\sqrt[3]{a}$, is positive. In Example 2, we first found the $g(x)$ function was

$$\frac{1}{\left(\sqrt[3]{x}\right)^2 + \left(\sqrt[3]{x}\right)\left(\sqrt[3]{a}\right) + \left(\sqrt[3]{a}\right)^2}$$

and then we decided we could drop the first two positive terms in the denominator and just use $\dfrac{1}{\sqrt[3]{a^2}}$ as the upper bound B for $|g(x)|$. Dividing by a fraction is accomplished by multiplying by its reciprocal, so we came up with $\epsilon\left(\sqrt[3]{a^2}\right)$ as our second choice for δ in Example 2.

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