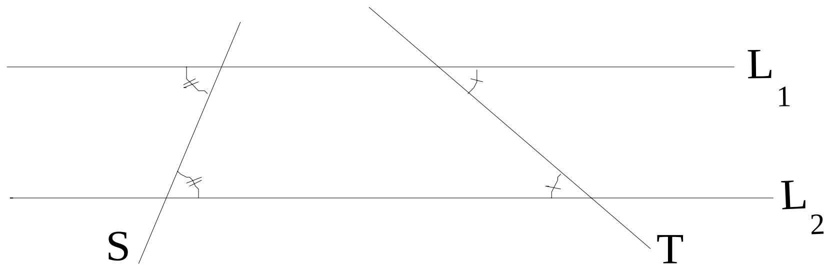
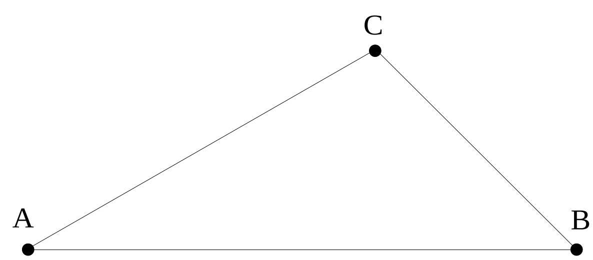
A Brief Review of Plane Geometry

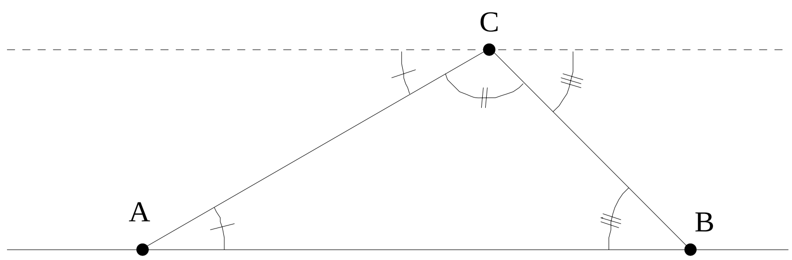
A fundamental property from geometry states that when two parallel lines are crossed by a third line, then pairs of corresponding angles are equal in measure. In the figure below, lines L_1 and L_2 are parallel and they are crossed by the two different lines S and T that are called ***transversal lines***. Pairwise, the angles that are ***alternate interior angles*** for each of the S and T lines are equal in measure, as indicated by the two sets of markings in the figure below. Other pairs of angles that look equal are also equal.



This fundamental property of parallel lines is responsible for the fact that the sum of the three angles of a triangle is 180°. To understand why the sum of the angles of a triangle is 180°, consider $\triangle ABC$ below.



We will extend the base side AB in both the left and right directions and we will draw a new line through point C so that this new line is parallel to the base line AB. See the next figure.

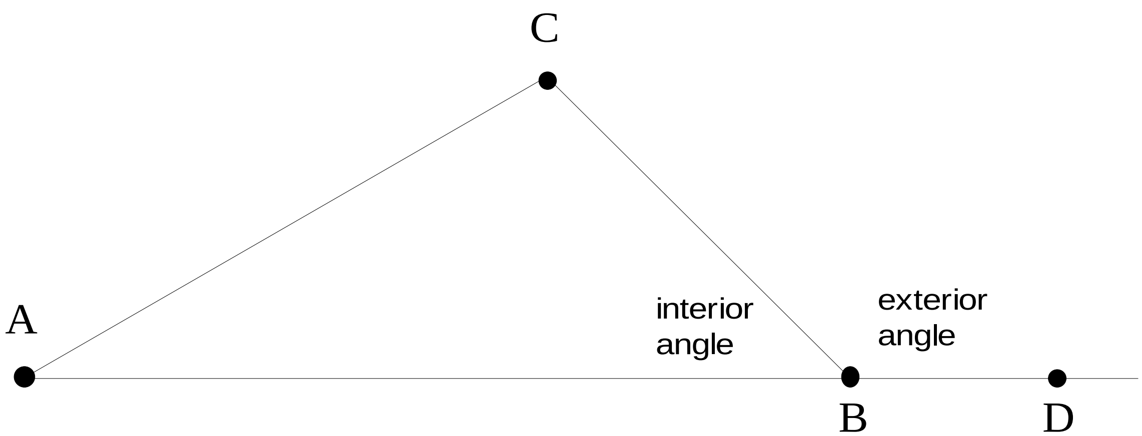


The three angles marked in the above figure at point C add up to 180° because they form the straight angle of the new line through C. But two of the angles at C are the same as the angles marked at points A and B in the triangle. The sameness of these pairs of angles is established by first considering the line AC as one transversal, and then considering the line BC as another transversal and labeling the alternate interior angles associated with these two transversals. Note that the three marked angles, $\angle A$, $\angle B$, and $\angle C$ are the three angles of the original triangle. So the measures of these three angles must sum to 180°.

$$m(\angle A) + m(\angle B) + m(\angle C) = 180^\circ.$$

Since the sum of the three angles of a triangle is 180°, if only two of the angles in a triangle are known, the third can be found. Two angles will be called ***supplementary*** if their measures add up to 180°. Two angles will be called ***complementary*** if their measures add up to 90°.

An ***exterior angle*** of a triangle is an angle formed by extending a side of the triangle. There are two exterior angles at each vertex of a triangle and they are supplementary with the interior angle at that vertex. In the figure below we have extended side AB to the right so that it contains point D.

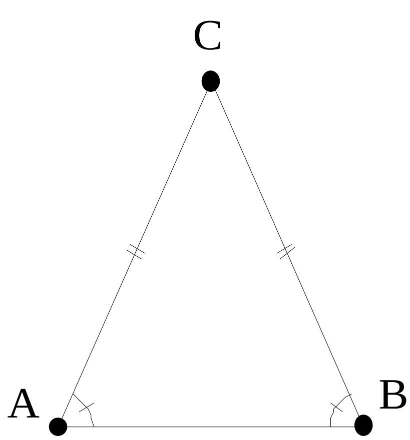


An exterior angle of a triangle is equal in measure to the sum of the two remote interior angles. The reason this is true is because the sum of the interior angle and the exterior angle is 180° and that is the sum of the three angles in the triangle. $m(\angle B) + m(\angle CBD) = m(\angle A) + m(\angle B) + m(\angle C)$. After subtracting $m(\angle B)$ from both sides of this equation we have:

$$m(\angle CBD) = m(\angle A) + m(\angle C)$$

An ***isosceles triangle*** is a triangle with two congruent sides. In this case the two angles opposite those congruent sides are also equal in measure. On the other hand, if two angles of a triangle are congruent, then the triangle is isosceles with the sides opposite those angles being equal.

An isosceles triangle:

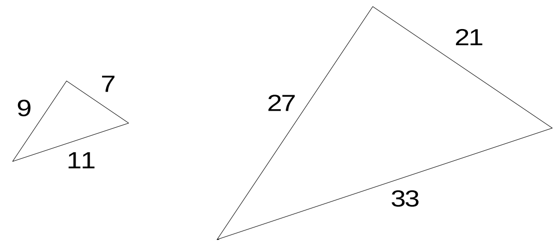


If $AC = BC$ then $m\angle A = m\angle B$.

If $m\angle A = m\angle B$ then $AC = BC$.

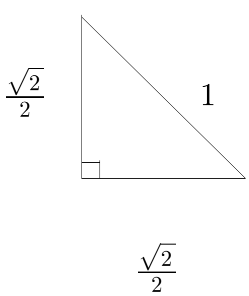
An ***equilateral triangle*** has all three sides equal in length. An ***equiangular triangle*** has all three angles equal in measure. Every equilateral triangle is equiangular and vice versa.

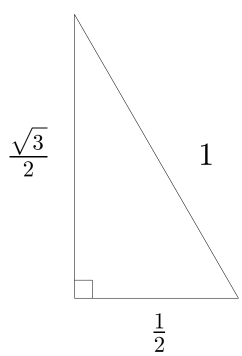
Congruent triangles have the same shape and size, whereas ***similar triangles*** need only have the same shape. For both congruent and similar triangles, all three angles of one triangle measure the same as the three corresponding angles of the other triangle. If only two angles of one triangle are congruent to two angles of the other triangle, then those two triangles are similar. What is important about similar triangles is that ratios of corresponding sides are equal.



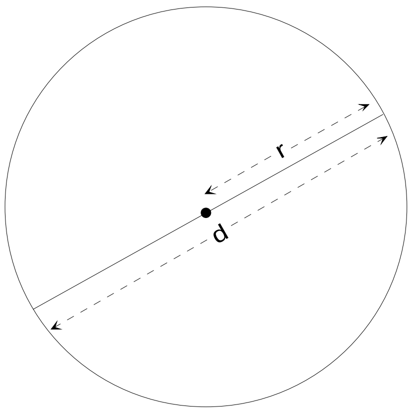
$$\frac{9}{27} = \frac{7}{21} = \frac{11}{33}$$

One major topic in trigonometry is computing angles and sides in triangles. In a ***30-60-90 right triangle*** the length of the shorter leg is one half of the hypotenuse and the length of the longer leg is $\sqrt{3}$ times the shorter leg. In a ***45-45-90 right triangle*** the lengths of the legs are always $\frac{\sqrt{2}}{2}$ times the length of the hypotenuse. The following are two standard triangles that we will use. The hypotenuse in each measures 1 unit. The lengths of the sides are determined by the relationships between the sides and the ***Pythagorean Theorem***. Memorize these! Triangles similar to these will have the same relationships between their sides and the measurements will simply be multiples of these standard measurements.





A circle is the set of all points equidistant from a given point.

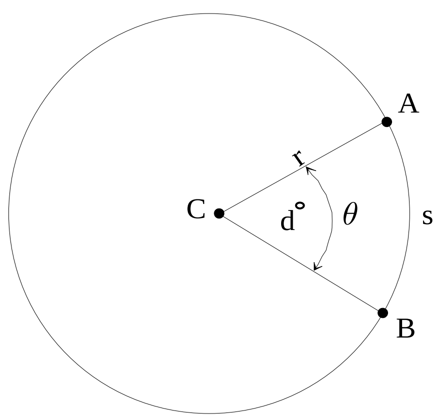


$d =$ diameter $\quad r =$ radius $\quad C =$ circumference $\quad A =$ area

$d = 2r \quad r = \frac{d}{2} \quad C = 2\pi r = \pi d \quad A = \pi r^2$

$$\pi = \frac{C}{d}$$

Note that the arc from A to B on the circle below can be measured in either of two ways. One way is to measure the number of degrees in the arc by measuring the central angle. The other way is to measure the actual curved distance from point A to point B. In the figure below the curved distance is labeled as s and is called the ***arclength***.



The following formula should be memorized.

$$\frac{d^\circ}{360^\circ} = \frac{\theta\, radians}{2\pi\, radians} = \frac{s}{2\pi r} = \frac{area\, of\ sector\ ABC}{\pi r^2}$$

Using the above formula we can derive the following two equations in which the angle θ must be measured in radians.

$$arclength\ s = r\theta$$

$$area\, of\ sector\ ABC = \tfrac{1}{2} r^2 \theta$$