A Method for Establishing $ln(x)$ and Computing e

Let a be a fixed positive constant where $a > 1$. Consider $\int_1^a \frac{1}{t} dt$. We wish to approximate this integral by using a Riemann sum in which the $n+1$ partition points for the closed interval $[1, a]$ form a geometric sequence, starting with $x_0 = 1$ and ending with $x_n = a$. The partition in this case is a non-uniform one, but as we shall show, it leads to a very interesting and succinct Riemann sum.

If x_i denotes any partition point and if we let $r > 0$ denote the common ratio of the geometric sequence, then we know $x_i = x_0 \cdot r^i$ where i is an integer between 0 and n inclusive. In particular we know $x_n = a = x_0 \cdot r^n = 1 \cdot r^n = r^n$. Using this equation we can explicitly solve for r and arrive at $r = \sqrt[n]{a} = a^{\frac{1}{n}}$. When $a > 1$ then $r > 1$.

We can consider two Riemann sum approximations for the above integral, the first where the function sample points are the left endpoints of each subinterval of the partition. In this case we have

$$\int_1^a \frac{1}{t} dt \approx \sum_{i=1}^{n} \frac{1}{x_{i-1}} \cdot [\, x_i - x_{i-1}] = \sum_{i=1}^{n} \left[\frac{x_i}{x_{i-1}} - \frac{x_{i-1}}{x_{i-1}} \right] = \sum_{i=1}^{n} [\, r - 1] = n \cdot [\, r - 1] = n \cdot \left[a^{\frac{1}{n}} - 1 \right]$$

After dividing both sides of this approximation by n we arrive at

$$\frac{\int_1^a \frac{1}{t} dt}{n} \approx a^{\frac{1}{n}} - 1 \quad \text{or} \quad 1 + \frac{\int_1^a \frac{1}{t} dt}{n} \approx a^{\frac{1}{n}}$$

Next, raise both sides of the last approximation to the n^{th} power.

$$\left[1 + \frac{\int_1^a \frac{1}{t} dt}{n} \right]^n \approx \left[a^{\frac{1}{n}} \right]^n = a^{\frac{n}{n}} = a$$

Then assuming a is the unique number such that $\int_1^a \frac{1}{t} dt = 1$ we have $\left[1 + \frac{1}{n} \right]^n \approx a$.

Finally we note that all of the above approximations depend on the Riemann sum approximation for the integral. We can make the integral approximation exact by taking the limit as $n \to \infty$. Thus we have the equation:

$$\lim_{n \to \infty} \left[1 + \frac{1}{n} \right]^n \;=\; \lim_{n \to \infty} a \;=\; a \tag{1}$$

Since the limit on the left yields the familiar decimal value $2.7182818\cdots$ that we know as e, what we have really shown is that $a = e$ when a ***is defined to be the number such that*** $\int_1^a \frac{1}{t} dt = 1$.

The above argument may also be made where instead of sampling the function $\frac{1}{t}$ at the left endpoint of each subinterval we sample $\frac{1}{t}$ at the right endpoint instead. The left-endpoint sum is really an upper Riemann sum whereas the right-endpoint sum is really a lower Riemann sum. The following shows essentially the same calculations using the right-endpoint Riemann sum. The geometric ratio r remains the same as above, but in this case we need to know $\frac{1}{r} = r^{-1} = a^{\frac{-1}{n}}$.

$$\int_1^a \frac{1}{t}dt \approx \sum_{i=1}^{n} \frac{1}{x_i} \cdot [\,x_i - x_{i-1}] = \sum_{i=1}^{n} \left[\frac{x_i}{x_i} - \frac{x_{i-1}}{x_i}\right] = \sum_{i=1}^{n} \left[1 - \frac{1}{r}\right] = n \cdot \left[1 - \frac{1}{r}\right] = n \cdot \left[1 - a^{\frac{-1}{n}}\right]$$

Next we divide both sides by n and solve for $a^{\frac{-1}{n}}$ in terms of the integral.

$$\frac{\int_1^a \frac{1}{t}dt}{n} \approx 1 - a^{\frac{-1}{n}} \quad \text{or} \quad a^{\frac{-1}{n}} \approx 1 - \frac{\int_1^a \frac{1}{t}dt}{n}$$

If we raise both sides of this last approximation to the n^{th} power then we can write

$$\left[a^{\frac{-1}{n}}\right]^n \approx \left[1 + \frac{-\int_1^a \frac{1}{t}dt}{n}\right]^n$$

Again we assume a is the unique value such that $\int_1^a \frac{1}{t}dt = 1$. We can now write

$$a^{\frac{-n}{n}} = a^{-1} \approx \left[1 + \frac{-1}{n}\right]^n$$

Taking the limit as $n \to \infty$ yields $a^{-1} = e^{-1}$ which is equivalent to showing that $a = e$.

We assume the reader is familiar with the fact that $e^x = \lim_{n\to\infty} \left[1 + \frac{x}{n}\right]^n$. (2)

A final remark is that using the first argument above, if we ***define*** $f(a) = \int_1^a \frac{1}{t}dt$, then since $a \approx \left[1 + \frac{f(a)}{n}\right]^n$, after taking the limit as $n \to \infty$, we get $a = e^{f(a)}$. This leads to determining that $f(x)$ is the inverse function of e^x and so we do **NOT** have to define $ln(x) = \int_1^x \frac{1}{t}dt$. Instead, we can begin the study of the transcendental functions by defining e and the exponential function e^x first, using limit expressions like those in (1) and (2) above, and then later we can derive the equivalent integral form for the natural logarithm function.

We can also estimate the Riemann sum for $\int_a^1 \frac{1}{t}dt$ where $0 < a < 1$, using arguments similar to those aleady given above. For this choice of a, the partition points are such that $x_0 = a$ and $x_n = 1$. We then have $1 = ar^n$ so that $r^n = \frac{1}{a}$ and this implies the geometric ratio $r = \sqrt[n]{\frac{1}{a}} = \sqrt[n]{a^{-1}} = a^{-\frac{1}{n}}$. We also know $r > 1$ because $\frac{1}{a} > 1$. Sampling the function at the left endpoint of each subinterval we have

$$\int_a^1 \frac{1}{t}dt \approx \sum_{i=1}^{n} \frac{1}{x_{i-1}} \cdot [\, x_i - x_{i-1}] = \sum_{i=1}^{n} \left[\frac{x_i}{x_{i-1}} - \frac{x_{i-1}}{x_{i-1}} \right] = \sum_{i=1}^{n} [\, r - 1] = n \cdot [\, r - 1] = n \cdot \left[a^{-\frac{1}{n}} - 1 \right]$$

After dividing both sides of this approximation by n we arrive at

$$\frac{\int_a^1 \frac{1}{t}dt}{n} \approx a^{-\frac{1}{n}} - 1 \quad \text{or} \quad 1 + \frac{\int_a^1 \frac{1}{t}dt}{n} \approx a^{-\frac{1}{n}}$$

Next, raise both sides of the last approximation to the n^{th} power.

$$\left[1 + \frac{\int_a^1 \frac{1}{t}dt}{n} \right]^n \approx \left[a^{-\frac{1}{n}} \right]^n = a^{-\frac{n}{n}} = a^{-1}.$$

After taking the limit as $n \to \infty$ and using (2) we have $e^{\int_a^1 \frac{1}{t}dt} = a^{-1}$. Using logs to solve for the exponent on e we have $\int_a^1 \frac{1}{t}dt = ln(a^{-1})$. $\int_a^1 \frac{1}{t}dt = -ln(a)$. $-\int_a^1 \frac{1}{t}dt = \int_1^a \frac{1}{t}dt = ln(a)$.

Thus we can establish whether $0 < a < 1$ or $a > 1$ that $ln(a) = \int_1^a \frac{1}{t}dt$.

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