Increments/Differentials/Tangent Planes/Differentiability

Example 1: This example shows that for a given surface, it is possible that both partial derivatives of the function may exist at a point, and yet the function surface may not even be continuous at that same point.

$$\text{Let } f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } x \neq 0 \text{ or } \; y \neq 0 \\ 0 & \text{if } x = 0 \text{ and } y = 0 \end{cases}$$

With a little work we can show that $f_x(0,0) = 0$ and $f_y(0,0) = 0$ so that both partial derivatives exist at the origin. However, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ D.N.E so that $f(x,y)$ is NOT continuous at the origin point. [Consider approaching $(0,0)$ along the lines $y = -x$ and $y = x$].

If $z = f(x,y)$ and if x changes by a small amount Δx at the same time that y changes by a small amount Δy then we can calculate the exact change in z, call it Δz by writing:

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x,y)$$

Assume we are given a function in the form $z = f(x,y)$. This equation represents a surface in 3-dimensional space. Assume the point (a,b) is a fixed interior point in the domain of $f(x,y)$. In other words, assume there exists a small circle with (a,b) at its center such that $f(x,y)$ is defined for all points (x,y) near (a,b). Now assuming $f_x(a,b)$ and $f_y(a,b)$ both exist we define another function

$$A(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)\cdot(x-a) + \frac{\partial f}{\partial y}(a,b)\cdot(y-b).$$

$A(x,y)$ has five simple but interesting and consequential properties.

1. The graph of $z = A(x,y)$ is a 3-dimensional surface that is a flat plane.

2. $A(a,b) = f(a,b)$ so that the point $P(a,b,f(a,b))$ is a point common to the original function surface $z = f(x,y)$ as well as the plane surface $z = A(x,y)$.

3. $\left.\dfrac{\partial A(x,y)}{\partial x}\right|_{(x,y)=(a,b)} = \dfrac{\partial A}{\partial x}(a,b) = \dfrac{\partial f}{\partial x}(a,b)$. The plane surface partial derivative matches the original function's partial derivative at the point (a,b) (at least in the x-direction as $y = b$ is held constant.).

4. $\left.\dfrac{\partial A(x,y)}{\partial y}\right|_{(x,y)=(a,b)} = \dfrac{\partial A}{\partial y}(a,b) = \dfrac{\partial f}{\partial y}(a,b)$. The plane surface partial derivative matches the original function's partial derivative at (a,b) (at least in the y-direction as $x = a$ is held constant).

5. When the point (x,y) is near (a,b) then $A(x,y)$ is usually (but NOT necessarily!) a good approximation to $f(x,y)$. The goodness of the approximation depends on the properties of $f(x,y)$'s partial derivatives near (a,b), with continuity being a decisive property as will be shown.

For any function $z = f(x, y)$ we formally define the differential $dz = \frac{\partial f}{\partial x}(x, y) \cdot dx + \frac{\partial f}{\partial y}(x, y) \cdot dy$ where also by definition, the differentials of the two independent variables x and y are defined by $dx = \Delta x$ and $dy = \Delta y$. In other words, if (x, y) is an initial point in the domain of $f(x, y)$ and if we change to a new point $(x + \Delta x, y + \Delta y)$ then the differential dz should give the approximate change in the z coordinate of the surface provided both Δx and Δy are small and $f(x, y)$ is well-behaved.

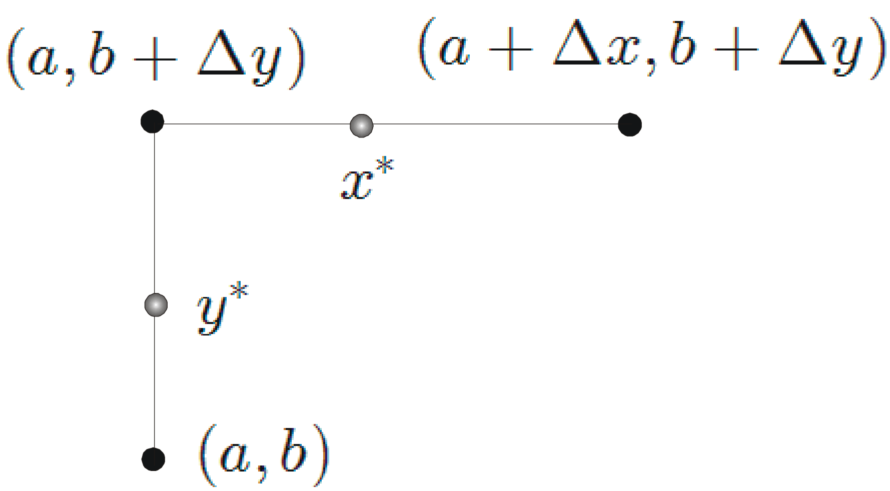
Theorem. Let $f(x, y)$ be a function of two variables. Assume (a, b) is a point in the domain of f where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both exist at all points in some small circular neighborhood with the center point (a, b). Furthermore, assume $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous at the point (a, b). Then there exist two functions $g(\Delta x, \Delta y)$ and $h(\Delta x, \Delta y)$ such that

$$\Delta z = \frac{\partial f}{\partial x}(a, b) \cdot \Delta x + \frac{\partial f}{\partial y}(a, b) \cdot \Delta y + g(\Delta x, \Delta y) \cdot \Delta x + h(\Delta x, \Delta y) \cdot \Delta y.$$

Moreover, the two functions $g(\Delta x, \Delta y)$ and $h(\Delta x, \Delta y)$ have the special limit properties that

$$\lim_{(\Delta x, \Delta y) \to (0,0)} g(\Delta x, \Delta y) = 0 \quad \text{and} \quad \lim_{(\Delta x, \Delta y) \to (0,0)} h(\Delta x, \Delta y) = 0.$$

Proof: Consider the following figure which shows points in the domain for $f(x, y)$ near the point (a, b).



The function $f(x, b + \Delta y)$ is a continuous function of the single variable x for all x over the closed x-interval $[a, a + \Delta x]$ and it is differentiable for all x over the open x-interval $(a, a + \Delta x)$ so that we can apply the Mean Value Theorem for a function of one variable (namely x). There exists a point $x^* \in (a, a + \Delta x)$ such that

$$f(a + \Delta x, b + \Delta y) - f(a, b + \Delta y) = \frac{\partial f}{\partial x}(x^*, b + \Delta y) \cdot \Delta x.$$

Similarly, the function $f(a, y)$ is a continuous function of y over the closed y-interval $[b, b + \Delta y]$ and it is differentiable over the open y-interval $(b, b + \Delta y)$ so that we can apply the Mean Value Theorem for a function of one variable (namely y). There exists a point $y^* \in (b, b + \Delta y)$ such that

$$f(a, b + \Delta y) - f(a, b) = \frac{\partial f}{\partial y}(a, y^*) \cdot \Delta y.$$

Now we can write

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

$$= [f(a + \Delta x, b + \Delta y) - f(a, b + \Delta y)] + [f(a, b + \Delta y) - f(a, b)]$$

$$= \left[\frac{\partial f}{\partial x}(x^*, b + \Delta y) \cdot \Delta x\right] + \left[\frac{\partial f}{\partial y}(a, y^*) \cdot \Delta y\right]$$

$$= \left[\frac{\partial f}{\partial x}(a, b) \cdot \Delta x - \frac{\partial f}{\partial x}(a, b) \cdot \Delta x + \frac{\partial f}{\partial x}(x^*, b + \Delta y) \cdot \Delta x\right] +$$

$$\left[\frac{\partial f}{\partial y}(a, b) \cdot \Delta y - \frac{\partial f}{\partial y}(a, b) \cdot \Delta y + \frac{\partial f}{\partial y}(a, y^*) \cdot \Delta y\right]$$

$$= \frac{\partial f}{\partial x}(a, b) \cdot \Delta x + \frac{\partial f}{\partial y}(a, b) \cdot \Delta y + \left[- \frac{\partial f}{\partial x}(a, b) \cdot \Delta x + \frac{\partial f}{\partial x}(x^*, b + \Delta y) \cdot \Delta x\right] +$$

$$\left[- \frac{\partial f}{\partial y}(a, b) \cdot \Delta y + \frac{\partial f}{\partial y}(a, y^*) \cdot \Delta y\right]$$

$$= \frac{\partial f}{\partial x}(a, b) \cdot \Delta x + \frac{\partial f}{\partial y}(a, b) \cdot \Delta y + \left[\frac{\partial f}{\partial x}(x^*, b + \Delta y) - \frac{\partial f}{\partial x}(a, b)\right] \Delta x +$$

$$\left[\frac{\partial f}{\partial y}(a, y^*) - \frac{\partial f}{\partial y}(a, b)\right] \Delta y$$

Now we let $g(\Delta x . \Delta y) = \left[\frac{\partial f}{\partial x}(x^*, b + \Delta y) - \frac{\partial f}{\partial x}(a, b)\right]$ and we let

$$h(\Delta x, \Delta y) = \left[\frac{\partial f}{\partial y}(a, y^*) - \frac{\partial f}{\partial y}(a, b)\right].$$

Then g and h are really functions of Δx and Δy because both x^* and y^* depend on Δx and Δy.

Note that as $\Delta x \to 0,\ x^* \to a$. Also, as $\Delta y \to 0,\ y^* \to b$.

Moreover, $\displaystyle \lim_{(\Delta x,\Delta y)\to(0,0)} \left[\frac{\partial f}{\partial x}(x^*, b+\Delta y) - \frac{\partial f}{\partial x}(a,b)\right] = \frac{\partial f}{\partial x}(a,b) - \frac{\partial f}{\partial x}(a,b) = 0$

and $\displaystyle \lim_{(\Delta x,\Delta y)\to(0,0)} \left[\frac{\partial f}{\partial y}(a, y^*) - \frac{\partial f}{\partial y}(a,b)\right] = \frac{\partial f}{\partial y}(a,b) - \frac{\partial f}{\partial y}(a,b) = 0.$

It is important to note that in calculating these last two limits we used the continuity of the partial derivative functions at the point (a,b) by saying the function value of the limit is the limit of the function values.

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \left[\frac{\partial f}{\partial x}(x^*, b+\Delta y)\right] = \frac{\partial f}{\partial x}(a,b) \ \text{ and } \ \lim_{(\Delta x,\Delta y)\to(0,0)} \left[\frac{\partial f}{\partial y}(a, y^*)\right] = \frac{\partial f}{\partial y}(a,b)$$

Corollary 1. Let $f(x,y)$ be a function of two variables. Assume (a,b) is a point in the domain of f where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both exist at all points in some small circular neighborhood with the center point (a,b). Furthermore, assume $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous at the point (a,b).

Let $A(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)\cdot(x-a) + \frac{\partial f}{\partial y}(a,b)\cdot(y-b)$. Then when $x = a + \Delta x$ and when $y = b + \Delta y$ we have

$$\left|\frac{f(x,y) - A(x,y)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}\right| = \left|\frac{f(x,y) - f(a,b) - \frac{\partial f}{\partial x}(a,b)\cdot\Delta x - \frac{\partial f}{\partial y}(a,b)\cdot\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}\right| =$$

$$\left|\frac{g(\Delta x,\Delta y)\cdot\Delta x + h(\Delta x,\Delta y)\cdot\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}\right| \le \left|\frac{g(\Delta x,\Delta y)\cdot\Delta x}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}\right| + \left|\frac{h(\Delta x,\Delta y)\cdot\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}\right|$$

$$\le |g(\Delta x,\Delta y)|\cdot\frac{|\Delta x|}{\sqrt{\Delta x^2 + \Delta y^2}} + |h(\Delta x,\Delta y)|\cdot\frac{|\Delta y|}{\sqrt{\Delta x^2 + \Delta y^2}}$$

$\le |g(\Delta x,\Delta y)| + |h(\Delta x,\Delta y)|$ and this last sum approaches 0 as $(\Delta x,\Delta y)\to(0,0)$.

This means $A(x,y)$ can approximate $f(x,y)$ to any desired degree of precision in any direction from the point (a,b) as long as (x,y) is near (a,b).

Definition Differentiable. Let $f(x,y)$ be a function of two variables. Assume (a,b) is a point in the domain of f. We say $f(x,y)$ is differentiable at (a,b) if

1) $f(x,y)$ is defined within a small circular neighborhood with the center point (a,b).

2) $\dfrac{\partial f}{\partial x}(a.b)$ exists.

3) $\dfrac{\partial f}{\partial y}(a,b)$ exists.

4) There exist functions $g(\Delta x, \Delta y)$ and $h(\Delta x, \Delta y)$ such that

$$\Delta z = \frac{\partial f}{\partial x}(a,b) \cdot \Delta x + \frac{\partial f}{\partial y}(a,b) \cdot \Delta y + g(\Delta x, \Delta y) \cdot \Delta x + h(\Delta x, \Delta y) \cdot \Delta y.$$

5) Moreover, the two functions $g(\Delta x, \Delta y)$ and $h(\Delta x, \Delta y)$ have the special limit properties that

$$\lim_{(\Delta x, \Delta y) \to (0,0)} g(\Delta x, \Delta y) = 0 \quad \text{and} \qquad \lim_{(\Delta x, \Delta y) \to (0,0)} h(\Delta x, \Delta y) = 0.$$

Definition Tangent Plane. Let $f(x,y)$ be a function of two variables. If $f(x,y)$ is differentiable at (a,b) then the plane given by the equation:

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b) \cdot (x-a) + \frac{\partial f}{\partial y}(a,b) \cdot (y-b)$$

is called the tangent plane to the surface $z = f(x,y)$ at the point (a,b).

Theorem. Let $f(x,y)$ be a function of two variables.
If $f(x,y)$ is differentiable at (a,b) then $f(x,y)$ is continuous at (a,b).

Proof: We need only show $\displaystyle\lim_{(\Delta x, \Delta y) \to (0,0)} f(a+\Delta x, b+\Delta y) = f(a,b)$.

However, this is easy since we know $f(a+\Delta x, b+\Delta y) - f(a,b) = \Delta z =$

$$\frac{\partial f}{\partial x}(a,b) \cdot \Delta x + \frac{\partial f}{\partial y}(a,b) \cdot \Delta y + g(\Delta x, \Delta y) \cdot \Delta x + h(\Delta x, \Delta y) \cdot \Delta y.$$

Now just take the limit as $\Delta x \to 0$ and $\Delta y \to 0$ to establish that $\Delta z \to 0$.

Example 2: The function given by: $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } x \neq 0 \text{ or } y \neq 0 \\ 0 & \text{if } x = 0 \text{ and } y = 0 \end{cases}$

is NOT differentiable at the point $(0,0)$ because it is NOT continuous at that same point.

Moreover, $A(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0)$ simplifies to

$A(x,y) = 0$. This means the plane $z = 0$ that is the xy-plane should be the approximating plane to $f(x,y)$. However, if we approach the origin point along the line $y = x$ then we know when x is near 0 that $f(x,x) = \dfrac{x^2}{x^2+x^2} = \dfrac{x^2}{2x^2} = \dfrac{1}{2}$. However, $f(0,0) = 0$. In this example,

$$|f(x,y) - A(x,y)| = |f(x,x) - A(x,x)| = \left|\frac{1}{2} - 0\right| = \frac{1}{2}.$$

This means $A(x,y)$ is a very poor approximation to $f(x,y)$ when x and y are the same even extremely small nonzero number.

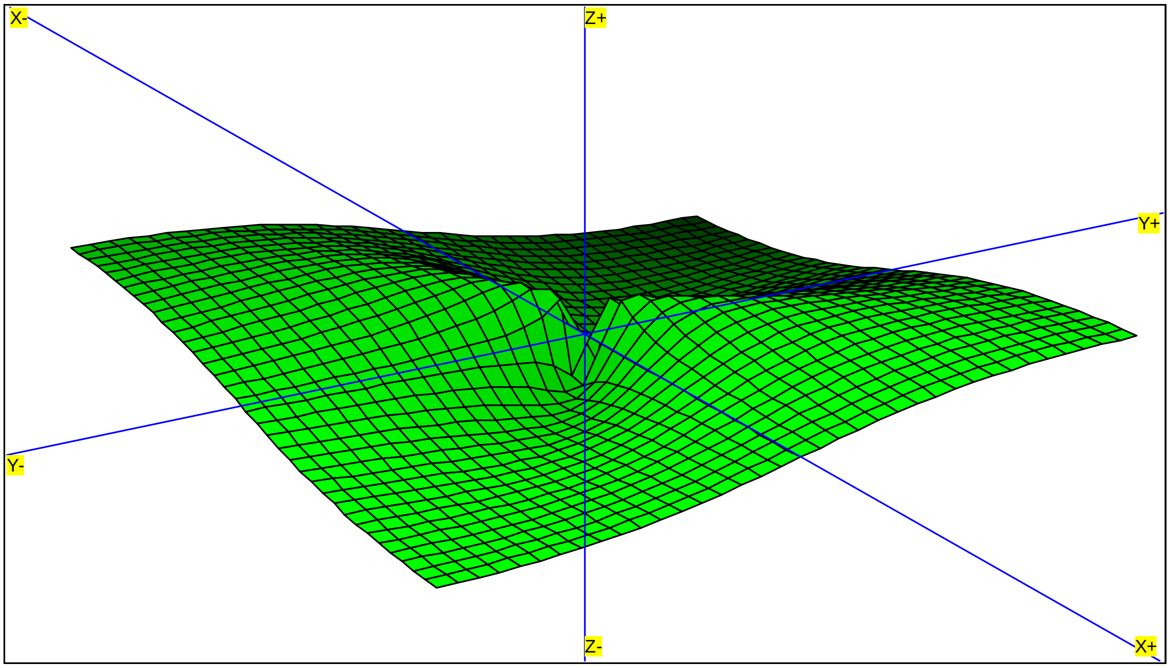
In fact, if we now try to calculate $\displaystyle\lim_{(\Delta x,\Delta y)\to(0,0)} \left| \frac{f(x,y) - A(x,y)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \right|$ where $\Delta y = \Delta x$ then we find

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \left| \frac{f(x,y) - A(x,y)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \right| = \lim_{\Delta x\to 0} \left| \frac{\frac{1}{2} - 0}{\sqrt{(\Delta x)^2 + (\Delta x)^2}} \right| =$$

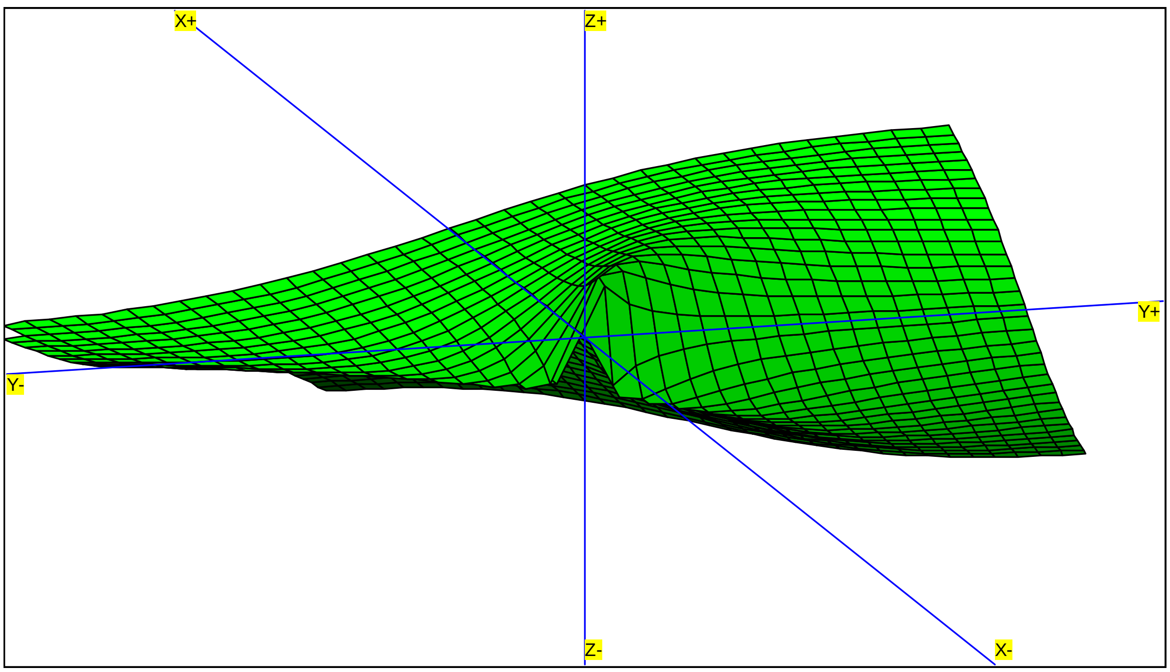
$$\lim_{\Delta x\to 0} \left| \frac{\frac{1}{2}}{\sqrt{2}\Delta x} \right| = \frac{1}{2\sqrt{2}} \cdot \lim_{\Delta x\to 0} \left| \frac{1}{\Delta x} \right| = +\infty.$$

This proves that $A(x,y)$ is truly an extremely poor approximation to $f(x,y)$ at the origin. The $f(x,y)$ function in this example is the same one given in Example 1 and we know that $f(x,y)$ is neither continuous nor differentiable at $(0,0)$.

The two figures on the next page show this surface from two different points of view. When looking at these two figures you have to imagine that there is no gap in the surface near the origin. No computer can make an accurate graph by dividing the surface into small surface patches, even when the size of those patches becomes very small. So the next two figures are inaccurate near the origin point, but if your mind can fill in the two gaps shown in the next two figures, then you will be able to imagine what the true surface looks like.



The graph of $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } x \neq 0 \text{ or } y \neq 0 \\ 0 & \text{if } x = 0 \text{ and } y = 0 \end{cases}$ as seen from above the xy-plane.



The graph of $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } x \neq 0 \text{ or } y \neq 0 \\ 0 & \text{if } x = 0 \text{ and } y = 0 \end{cases}$ as seen from below the xy-plane.

Corollary 2. Let $f(x, y)$ be a function of two variables. Assume (a, b) is a point in the domain of f where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both exist at all points in some small circular neighborhood whose center point is the point (a, b). Furthermore, assume $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous at the point (a, b). Then $f(x, y)$ is differentiable at (a, b).