Some First Semester Calculus Definitions and Theorems

by

John Kennedy
Santa Monica College
1900 Pico Blvd.
Santa Monica, CA 90405

`rkennedy@ix.netcom.com`

Some First Semester Calculus Definitions and Theorems

These pages contain just a few theorems and definitions and examples from the first course in calculus. Not all the standard definitions and theorems are included here, but the more important results and some special techniques and generalizations are represented. An elementary knowledge of limits is required to read and understand these notes. A couple of proofs may require a little extra effort by the reader. These notes are for math majors and those interested in rigorous definitions and proofs as found in real analysis. It might be helpful for you to have completed the 2nd course in calculus before you try to read these notes.

The biggest challenges are understanding the Completeness Axiom for the real number system, the Monotonic Convergence Theorem, the Nested Intervals Property, and the Bolzano-Weierstrass Theorem. These theorems are not always presented in either the first or the second semester of calculus. But once these theorems are mastered, the proofs of the two most important theorems for continuous functions, namely the Extreme Value Theorem and the Intermediate Value Theorem will be relatively easy.

The Nested Interval Property is the main theoretical result that lies behind the Method of Successive Bisections that is an essential computer technique for solving equations. The Method of Successive Bisections is presented in most modern calculus text books that never mention the Nested Interval property. Our main use of the Completeness Axiom and the Nested Interval Property is to prove the existence of certain values related to limits and continuous functions. In fact, our proofs are constructive in the sense that they tell exactly how to construct the existence of certain values. While alternative indirect proofs may appear to be shorter, they are less intuitive and lack the virtue of being constructive and lack a direct connection to computer algorithms.

1. Definition. (Limit)

Let L be a finite number. To say that $L = \lim_{x \to a} f(x)$ means $^{\forall}\epsilon > 0$ $^{\exists}\delta{>}0$ such that whenever $0 < |x - a| < \delta$ then we must have $|f(x) - L| < \epsilon$.

2. **Theorem. (Limit Inequality)**

If $f(x) \leq g(x)$ for all x in an open interval I containing $x = a$, and if $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$ then $L \leq M$.

Proof: By contradiction. Assume $L > M$. Let $\epsilon = \dfrac{L - M}{2} > 0$. Then there exist δ_1 and δ_2 such that if $0 < |x - a| < \delta_1$ then $|f(x) - L| < \epsilon$ and if $0 < |x - a| < \delta_2$ then $|g(x) - M| < \epsilon$. Now let $\delta = min\{\delta_1, \delta_2\}$ and choose x_0 such that $x_0 \in I$ and $0 < |x_0 - a| < \delta$. Then clearly $x_0 \neq a$.
Moreover, $L - f(x_0) < \epsilon$ and $g(x_0) - M < \epsilon$ so adding these two inequalities we have the special inequality that $L - f(x_0) + g(x_0) - M < 2\epsilon$. Now $L - M = 2\epsilon$.
Thus our special inequality says $2\epsilon + [\,g(x_0) - f(x_0)\,] < 2\epsilon$ which in turn implies $g(x_0) - f(x_0) < 0$ or $g(x_0) < f(x_0)$. This contradicts that $f(x) \leq g(x)$ for all $x \in I$.

QED

3. **Theorem. (Squeeze or Sandwich Theorem)**

Let $f(x)$, $g(x)$, and $h(x)$ be any three functions that share a common domain that includes an open interval I that contains the point $x = a$. Assume for all $x \in I$ that $f(x) \leq g(x) \leq h(x)$ and assume all three functions have a limit as $x \to a$. If $\lim\limits_{x \to a} f(x) = L$ and $L = \lim\limits_{x \to a} h(x)$ then $\lim\limits_{x \to a} g(x) = L$.

Proof: Apply the above Limit Inequality Theorem 2. twice, first with $f(x)$ and $g(x)$ to conclude that $\lim\limits_{x \to a} g(x) \geq L$ and then a second time with $g(x)$ and $h(x)$ to conclude that $\lim\limits_{x \to a} g(x) \leq L$.
These two inequalities establish that $\lim\limits_{x \to a} g(x) = L$.

QED

4. **Remarks.**

Both the Limit Inequality Theorem 2. and the Squeeze Theorem 3. hold when all the limits are replaced by the same kind of a one-sided limit that can be either from the left or from the right.
The existence of $\lim\limits_{x \to a} g(x)$ is not a required hypothesis in the Squeeze Theorem 3. statement.
The inequality that $f(x) \leq g(x) \leq h(x)$ and the existence of the same limit for the $f(x)$ and $h(x)$ functions is all that is required to establish the existence of the limit for $g(x)$.
An alternative proof of Theorem 3. is as follows. Let $\epsilon > 0$ be given. Since $\lim\limits_{x \to a} f(x) = L$ there exists a δ_1 such that if $0 < |x - a| < \delta_1$ then $|f(x) - L| < \epsilon$. Since $\lim\limits_{x \to a} h(x) = L$ there exists a δ_2 such that if $0 < |x - a| < \delta_2$ then $|h(x) - L| < \epsilon$. Now let $\delta = min\{\delta_1, \delta_2\}$ and assume x is such that $0 < |x - a| < \delta$. Then $-\epsilon < f(x) - L$ and $h(x) - L < \epsilon$. Since $f(x) \leq g(x) \leq h(x)$ we know $f(x) - L \leq g(x) - L \leq h(x) - L$. So combining the above inequalitites we have $-\epsilon < f(x) - L \leq g(x) - L \leq h(x) - L < \epsilon$. This implies $|g(x) - L| < \epsilon$. So given any $\epsilon > 0$ we can find a δ such that if $0 < |x - a| < \delta$ then $g(x)$ is within ϵ of L. This establishes that $\lim\limits_{x \to a} g(x) = L$.

QED

5. Theorem. (Positive Limit)

If $\lim\limits_{x \to a} f(x) > 0$ then $^{\exists}\delta > 0$ such that if $0 < |x - a| < \delta$ then $f(x) > 0$.

Proof: Let $L = \lim\limits_{x \to a} f(x) > 0$. Choose $\epsilon = \frac{L}{2}$. Then there exists a δ>0 such that whenever $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$. This means $-\frac{L}{2} < f(x) - L < \frac{L}{2}$ which in turn implies that $0 < \frac{L}{2} < f(x) < \frac{3L}{2}$.

QED

6. Theorem. (Negative Limit)

If $\lim\limits_{x \to a} f(x) < 0$ then $^{\exists}\delta > 0$ such that if $0 < |x - a| < \delta$ then $f(x) < 0$.

Proof: Let $L = \lim\limits_{x \to a} f(x) < 0$. Choose $\epsilon = -\frac{L}{2} > 0$. Then there exists a δ>0 such that whenever $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$. This means $-\left(-\frac{L}{2}\right) < f(x) - L < -\frac{L}{2}$ which in turn implies that $\frac{3L}{2} < f(x) < \frac{L}{2} < 0$.

QED

7. Definition. Continuous Function At A Point

Let $f(x)$ be any function whose domain includes an open interval that contains the point $x = p$.
We say $f(x)$ is continuous at $x = p$ if the following three conditions hold.

1) $f(p)$ is defined.
2) $\lim\limits_{x \to p} f(x)$ exists.
3) $f(p) = \lim\limits_{x \to p} f(x)$.

8. Definition. Continuous Function Over An Open Interval

Let $f(x)$ be any function whose domain includes an open interval (a, b).
We say $f(x)$ is continuous over (a, b) if $f(x)$ is continuous at every point $p \in (a, b)$.

9. Definition. Continuous Function At An Endpoint

Let $f(x)$ be any function whose domain includes a closed interval $[\,a, b\,]$. We say $f(x)$ is continuous at the left endpoint a or at the right endpoint b if the two limits in the definition for continuity at a point are replaced by the appropriate one-sided limits. The two limits are from the right at the left endpoint $x = a$ and the two limits are from the left at the right endpoint $x = b$.

10. Theorem. (Continuous Function Sequence Limit)

Let $f(x)$ be continuous over an open interval (a, b) and let $\{a_n\}$ be any convergent sequence whose values and limit are in (a, b). Then $f\left(\lim\limits_{n\to\infty} a_n\right) = \lim\limits_{n\to\infty} f(a_n)$.

Proof: Let $c = \lim\limits_{n\to\infty} a_n \in (a, b)$. Claim that $f(c) = \lim\limits_{n\to\infty} f(a_n)$.

Let $\epsilon > 0$ be given. Since $f(x)$ is continuous at $x = c$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - f(c)| < \epsilon$. Now since $c = \lim\limits_{n\to\infty} a_n$ and $\delta > 0$ there exists an integer N such that if $n > N$ then $|a_n - c| < \delta$. So if we choose $n > N$ then $|a_n - c| < \delta$ which in turn implies $|f(a_n) - f(c)| < \epsilon$. We have shown for every $\epsilon > 0$, eventually $f(a_n)$ is within ϵ of $f(c)$.

This means $f(c) = \lim\limits_{n\to\infty} f(a_n)$.

QED

11. Remark.

Note that the above Sequence Limit Theorem 10. is closely related to the definition of continuity at a point in the sense that $f\left(\lim\limits_{x\to a} x\right) = \lim\limits_{x\to a} f(x)$.

12. Definition. Function Increasing Over An Interval

A function is called increasing over an interval I if whenever x_1 and x_2 are real numbers in I and $x_1 < x_2$ then we must have $f(x_1) < f(x_2)$. The interval I may be closed or half-open or may extend to infinity in either the left or right or both left and right directions.

13. Definition. Function Decreasing Over An Interval

A function is called decreasing over an interval I if whenever x_1 and x_2 are real numbers in I and $x_1 < x_2$ then we must have $f(x_1) > f(x_2)$. The interval I may be closed or half-open or may extend to infinity in either the left or right or both left and right directions.

14. Remark.

Constant functions are neither increasing nor decreasing over any kind of interval.

15. Definition. Local Maximum for a Function

Let $f(x)$ be defined over an open interval containing the point c. The point $(c, f(c))$ is called a local maximum for $f(x)$ if $f(x) \leq f(c)$ for all x in the domain of $f(x)$ near c.

16. Definition. Local Minimum for a Function

Let $f(x)$ be defined over an open interval containing the point c. The point $(c, f(c))$ is called a local minimum for $f(x)$ if $f(x) \geq f(c)$ for all x in the domain of $f(x)$ near c.

17. Definition. Absolute Maximum for a Function

The point $(c, f(c))$ is called an absolute maximum for $f(x)$ if $f(x) \leq f(c)$ for all x in the domain of $f(x)$.

18. Definition. Absolute Minimum for a Function

The point $(c, f(c))$ is called an absolute minimum for $f(x)$ if $f(x) \geq f(c)$ for all x in the domain of $f(x)$.

19. Remark.

Every point on the graph of $f(x) = k$ where k is a constant is a local maximum and a local minimum and an absolute maximum and an absolute minimum. Thus a function may have more than one absolute extrema or local extrema.

20. Axiom. (Completeness Axiom For Real Numbers)

Every nonempty set of real numbers that is bounded above has a least upper bound and every nonempty set of real numbers that is bounded below has a greatest lower bound.

21. Remark.

Note that the lower bound property is a consequence of the upper bound property if we just multiply each number in the set by -1. Thus a minimal axiom would just be the upper bound statement. In one form or another, the Completeness Axiom 20. is needed to explain an essential element of the real number system. Below, we will use the Completeness Axiom to prove a special property that is called the Nested Interval Property.

22. Theorem. (Monotonic Convergence)

1) If $\{a_n\}$ is an increasing sequence of numbers that is bounded above, then $\lim\limits_{n\to\infty} a_n$ exists.

2) If $\{b_n\}$ is a decreasing sequence of numbers that is bounded below, then $\lim\limits_{n\to\infty} b_n$ exists.

Proof: We only prove the increasing bounded above case because the proof of the decreasing bounded below case is similar. In fact, if the terms of either kind of sequence are multiplied by -1 the other kind of sequence is obtained.

Apply the Completeness Axiom 20. for real numbers to the set that is the range of the sequence $\{a_n\}$ and let L denote the least upper bound of that set. Claim that $\lim\limits_{n\to\infty} a_n = L$.

Let $\epsilon > 0$ be given. Then $L - \epsilon$ is less than the least upper bound of the sequence range. Claim that there exists an integer N such that $a_N > L - \epsilon$. For otherwise, if $a_n \leq L - \epsilon$ for all n, then $L - \epsilon$ would be a smaller upper bound for the sequence range as compared to L itself. This contradicts the fact that L is the least upper bound for the sequence range. So there must exist some N such that $a_N > L - \epsilon$. Since a_n is increasing, whenever $n > N$, we also have $a_n \geq a_N > L - \epsilon$. We know for all n, $a_n \leq L$ because L is an upper bound for the sequence and we know there exists an integer N such that whenever $n > N$ then $a_n - L > -\epsilon$. So if $n > N$ then $-\epsilon < a_n - L \leq 0$. So $|\,a_n - L| < |-\epsilon| = \epsilon$. We have shown $\lim\limits_{n\to\infty} a_n = L$.

QED

23. Theorem. (Nested Interval Property)

Let $\{I_n\}$ denote a sequence of closed intervals such that for all $n, I_{n+1} \subseteq I_n$ and such that $\lim\limits_{n\to\infty} |I_n| = 0$ where $|I_n|$ denotes the length of the interval I_n. Then $\bigcap\limits_{n=1}^{\infty} I_n = \{r\}$ for some real number r.

Proof: The Nested Interval Property and the Completeness Axiom 20. are equivalent to one another, but here we only show that the Completeness Axiom implies the Nested Interval Property. We need to prove both the existence and uniqueness of r. We let $I_n = [\,a_n, b_n\,]$ so that we may refer to the two sequences that are the left and right endpoints of each closed interval. The sequence $\{a_n\}$ is increasing and bounded above by b_1. Thus by the Monotonic Convergence Theorem 22. the sequence $\{a_n\}$ has a limit L_a that is the Least Upper Bound of the set of numbers $\{a_n\}$. The sequence $\{b_n\}$ is decreasing and bounded below by a_1. So again by the Monotonic Convergence Theorem 22. the sequence $\{b_n\}$ has a limit G_b that is the Greatest Lower Bound of the set of numbers $\{b_n\}$. Next we will prove that $L_a = G_b$. To do this we will show that if $L_a < G_b$ then we get a contradiction and we will show that if $L_a > G_b$ we also get a contradiction. First, temporarily assume that $L_a < G_b$. Let $\epsilon = G_b - L_a$. For all n we have $a_n \leq L_a < G_b \leq b_n$ which implies $b_n - a_n \geq G_b - L_a = \epsilon$. This contradicts the property that the difference between b_n and a_n can be made arbitrarily small by going out far enough in both sequences because $\lim\limits_{n\to\infty} |I_n| = 0$. Next, temporarily assume $L_a > G_b$. Let $\epsilon = \dfrac{L_a - G_b}{2}$ so that $G_b + 2\epsilon = L_a$ or $G_b + \epsilon = L_a - \epsilon$. Since L_a is the Least Upper Bound of the sequence $\{a_n\}$, there exists some subscript P such that $a_P > L_a - \epsilon$. For otherwise, if $a_n \leq L_a - \epsilon$ for all n, then $L_a - \epsilon$ would have been an upper bound for the sequence $\{a_n\}$ that would be smaller than the supposed Least Upper Bound, L_a. Also, since G_b is the Greatest Lower Bound of the set $\{b_n\}$ there exists some subscript Q such that $b_Q < G_b + \epsilon$. For otherwise, if $b_n \geq G_b + \epsilon$ for all n, then $G_b + \epsilon$ would be a lower bound that would be greater than the supposed Greatest Lower Bound, G_b. Finally, we can write $b_Q < G_b + \epsilon = L_a - \epsilon < a_P$. Now let M be any integer larger than the maximum of both P and Q. Then since $\{a_n\}$ is increasing and $\{b_n\}$ is decreasing we have $b_M \leq b_Q < a_P \leq a_M$. That $b_M < a_M$ contradicts the fact that for all n we should have $a_n \leq b_n$. With these two contradictions we have shown $L_a \geq G_b$ and $L_a \leq G_b$. This of course means $L_a = G_b$ and so we may let the real number r be such that $r = L_a = G_b$. Since r is an upper bound for $\{a_n\}$ we know for all n, $a_n \leq r$. Also, since r is a lower bound for $\{b_n\}$ we know for all n that $r \leq b_n$. These last two statements imply that for all n, $r \in [\,a_n, b_n\,]$ so $r \in \bigcap\limits_{n=1}^{\infty} I_n$. To prove r is the only number in this set we assume $r_2 \in \bigcap\limits_{n=1}^{\infty} I_n$ is a second element where for all n, $a_n \leq r_2 \leq b_n$. Then r_2 is both an upper bound for $\{a_n\}$ and a lower bound for $\{b_n\}$. Since r is the Least Upper Bound for $\{a_n\}$ we know $r \leq r_2$. Since r is the Greatest Lower Bound for $\{b_n\}$ we know $r_2 \leq r$. These two inequalities imply $r = r_2$. For the future we note that r has the special property that $\lim\limits_{n\to\infty} a_n = r = \lim\limits_{n\to\infty} b_n$.

QED

$\boxed{24.}$ Theorem. (Bolzano-Weierstrass)

a) If S denotes an infinite set of real numbers that is bounded then there exists a number L such that for every $\epsilon > 0,$ the set $T = \{x : L - \epsilon < x < L + \epsilon \text{ and } x \neq L\}$ has the property that $T \cap S \neq \emptyset$.

b) Let $\{x_n\}$ denote an infinite sequence of real numbers all contained within a closed interval $[\,a, b]$. Then there exists a subsequence $\{x_{m_k}\}$ and there exists a number $L \in [\,a, b]$ such that $L = \lim\limits_{k\to\infty} x_{m_k}$.

Proof: a) Since S is bounded there exists a number M such that $S \subset [-M, M]$. Let I_0 denote the interval $[-M, M]$. When I_0 is bisected, one of its halves must contain an infinite number of points of S, because otherwise, if both halves only contained a finite number of points of S then their union that is I_0 would only contain a finite number of points of S. Having defined I_j we will recursively define I_{j+1} to be that half of I_j such that I_{j+1} contains an infinite number of points of S. Note that the length of I_j is $\dfrac{M}{2^{j-1}}$. By the Nested Interval Property $\boxed{23.}$ we may let $L \in \bigcap\limits_{j=1}^{\infty} I_j$. Now let $\epsilon > 0$ be given. When j is sufficiently large, $\dfrac{M}{2^{j-1}} < \epsilon$. Now because no point of I_j is further away from L than ϵ, we must have $I_j \subset \{x : L - \epsilon < x < L + \epsilon\}$. Now consider the set $T = \{x : L - \epsilon < x < L + \epsilon \text{ and } x \neq L\}$. Since I_j contains an infinite number of points of S, we can choose one, different from L, and thus establish that $S \cap I_j \neq \emptyset$ which in turn implies $T \cap S \neq \emptyset$.

b) The set of values $\{x_n\}$ is infinite and is bounded. So we can apply part a) to conclude there exists a number L such that some x_n differs from L by no more than ϵ for every $\epsilon > 0$. In particular there exists an $m_1$such that $|L - x_{m_1}| < 1$. Next, there are an infinite number of $n's$ such that $|L - x_n| < \frac{1}{2}$. So we can choose m_2 such that $m_2 > m_1$ and $|L - x_{m_2}| < \frac{1}{2}$. Continuing in this same manner, there are an infinite number of $n's$ such that $|L - x_n| < \frac{1}{3}$. So we can choose m_3 such that $m_3 > m_2$ and $|L - x_{m_3}| < \frac{1}{3}$. So if we continue in this manner, for each positive integer k there exists an $m_k > m_{k-1}$ such that $|L - x_{m_k}| < \frac{1}{k}$. It is now clear that the subsequence $\{x_{m_k}\}$ has the property that $L = \lim\limits_{k\to\infty} x_{m_k}$.

QED

25. **Theorem. (Bounded Range Over a Closed Interval)**

Let $f(x)$ be continuous on a closed interval $[\,a, b\,]$. Then there exists a number M such that for all $x \in [\,a, b]$, $|f(x)| \leq M$. In other words, the range of $f(x)$ is bounded.

Proof: Consider the set $S = \{u : f(x)$ is bounded on the closed interval $[\,a, u]$ where $a < u \leq b.\}$ Since $f(x)$ is continuous at $x = a$, if we choose $\epsilon = 1$, then there exists a $\delta > 0$ such that whenever $x \in [\,a, b\,]$ and $|x - a| < \delta$ then $|f(x) - f(a)| < 1$. In other words, if $a \leq x \leq a + \frac{\delta}{2}$ then $-1 + f(a) < f(x) < 1 + f(a)$. This means $f(x)$ is bounded over the closed interval $[\,a, a + \frac{\delta}{2}]$ so that $a + \frac{\delta}{2} \in S$. We have established that S is nonempty. S is also bounded above by $u = b$. Applying the Completeness Axiom 20., the set S thus has a least upper bound that we denote by L. We will establish that $L = b$ by assuming $L < b$ and getting a contradiction.

First note that since $f(x)$ is continuous on $[\,a, b]$ is it continuous at $x = L$. Now choose $\epsilon = 1$ and get a δ such that if $x \in [\,a, b]$ and $|\,x - L| < \delta$ then $|f(x) - f(L)| < 1$. Then whenever $-\frac{\delta}{2} + L \leq x \leq L + \frac{\delta}{2}$ we must have $-1 + f(L) < f(x) < 1 + f(L)$. So $f(x)$ is bounded over the two closed intervals $[\,a, -\frac{\delta}{2} + L]$ and $[\,-\frac{\delta}{2} + L, L + \frac{\delta}{2}]$. By applying the maximum of the two bounds for these two intervals we know $f(x)$ is bounded over the interval $[\,a, L + \frac{\delta}{2}]$. This means $L + \frac{\delta}{2} \in S$ and this contradicts the fact that L was the least upper bound of S if $L < b$. However, if $L = b$ then the contradiction does not occur because all x values are first restricted to the closed interval $[\,a, b]$ and in all such cases $x \leq b$.

QED

26. Lemma. (Continuous Functions Can't Creep!)

Let $f(x)$ be continuous on a closed interval $[a, b]$. Then either there exists an $x^* \in [a, \frac{a+b}{2}]$ such that for all $z \in [\frac{a+b}{2}, b]$ $f(z) \le f(x^*)$ or there exists an $x^* \in [\frac{a+b}{2}, b]$ such that for all $z \in [a, \frac{a+b}{2}]$ $f(z) \le f(x^*)$. In other words, one half of $[a, b]$ contains an upper bound for all function values in the other half of $[a, b]$.

Proof: By contradiction. Assume $f(x)$ has the property that for each $x^* \in [a, \frac{a+b}{2}]$ there exists a $z \in [\frac{a+b}{2}, b]$ such that $f(z) > f(x^*)$ and for each $x^* \in [\frac{a+b}{2}, b]$ there exists a $z \in [a, \frac{a+b}{2}]$ such that $f(z) > f(x^*)$. We call this the **creeping property** for the function over the closed interval $[a, b]$. Our goal is to show that a continuous function defined over a closed interval does **NOT** have this property.

Let $S = \{f(x) : x \in [a, b]\}$. In other words, S is the range of the function. S is nonempty and by Theorem 25. S is bounded above. So the range set S has a least upper bound that we denote by L. For all $x \in [a, b]$ we must have $f(x) < L$. We cannot have $f(x) = L$ because otherwise, by the creeping property, we could find a z in the half of $[a, b]$ that does not contain x such that $f(z) > f(x) = L$ and this would contradict the fact that L is the least upper bound for the range of f.

Next we are going to construct a sequence of domain points in $[a, b]$. We let x_0^* be any starting point in the domain $[a, b]$. Whatever half of $[a, b]$ contains x_0^* there exists an x_1^* in the other half such that $f(x_1^*) > f(x_0^*)$. Then whatever half of $[a, b]$ contains x_1^* we apply the creeping property to find an x_2^* in the other half such that $f(x_2^*) > f(x_1^*) > f(x_0^*)$. We keep applying the creeping property to construct an infinite sequence of increasing range values $f(x_n^*)$. All values in this range sequence are bounded above by L and yet the range sequence is increasing. Thus by the Monotonic Convergence Theorem 22. the range sequence must converge to L.

$\lim\limits_{n\to\infty} f(x_n^*) = L$. Now the domain sequence $\{x_n^*\}$ is an infinite subset of $[a, b]$ and so by the Bolzano-Weierstrass Theorem 24. it must contain a convergent subsequence, call it $\{s_m^*\}$.

Let $s_0 = \lim\limits_{m\to\infty} s_m^*$. Then $s_0 \in [a, b]$ and we wish to consider the value $f(s_0)$.

By the Continuous Function Sequence Limit Theorem 10., $f(s_0) = f\left(\lim\limits_{m\to\infty} s_m^*\right) = \lim\limits_{m\to\infty} f(s_m^*) = \lim\limits_{n\to\infty} f(x_n^*) = L$.

Now if $f(x)$ has the creeping property, whatever half of $[a, b]$ contains s_0 we can find a z in the other half such that $f(z) > f(s_0) = L$. This contradicts the fact that L was the least upper bound of the range set S.

QED

27. Theorem. (Extreme Value Theorem)

Let $f(x)$ be continuous on a closed interval $[a, b]$. Then there exist numbers c and d in $[a, b]$ (not necessarily distinct and not necessarily unique) such that the point $(c, f(c))$ is an absolute minimum of $f(x)$ and the point $(d, f(d))$ is an absolute maximum of $f(x)$.

Proof: We only prove this for the case of an absolute maximum, since once that is established the minimum value can be proved by considering the function $-f(x)$. Any absolute minimum of any function $g(x)$ must be an absolute maximum of the function $-g(x)$ and vice versa.

We will create a sequence of nested closed intervals and a sequence of special points. The first interval is denoted by I_0 and we let $I_0 = [a, b]$. Having defined I_j, we recursively define I_{j+1} to be either the left or right half of I_j by the decision rule that I_{j+1} is the left half of I_j if the left half of I_j contains an x-value denoted by x_j^* such that $f(x_j^*) \geq f(z)$ for all z in the right half of I_j. Otherwise, if no such x_j^* value exists in the left half of I_j then I_{j+1} is chosen as the right half of I_j. Note that by Lemma 26. when I_{j+1} is chosen as the right half of I_j, then that right half contains an x-value that we also denote by x_j^* such that $f(x_j^*) \geq f(z)$ for all z in the left half of I_j. We will call the qualifying x_j^* number the *leader* for the I_j interval. In all cases we can find a qualifying x-value, say x_j^*, in one half of I_j such that $f(x_j^*) \geq f(z)$ for all z in the other half of I_j.

Our decision choice should always be possible and should never be ambiguous. For the case in which $f(x)$ is constant over the entire I_j interval we will always choose and continue to only choose the left subinterval as I_{j+1}. In selecting x_{j+1}^*, we choose such a value so that the $f(x_j^*)$ values form a non-decreasing sequence. $f\left(x_j^*\right) \leq f\left(x_{j+1}^*\right)$. Also, for all $z \in [a, b] - I_j$ we have $f(z) \leq f(x_j^*)$ where x_j^* is the leader for I_j.

Since each next interval is half the length of its predecessor, the closed intervals keep shrinking in size. As j increases, the set $[a, b] - I_j$ becomes larger and includes more of $[a, b]$.

By the Nested Interval Property, Theorem 23., we may let $x_0 = \bigcap_{j=1}^{\infty} I_j$. Then $x_0 = \lim_{j\to\infty} x_j^*$ because the points x_j^* always lie between the endpoints of each I_j interval and both end point sequences converge to the same limiting value, namely x_0. Claim that $f(x_0)$ is an absolute maximum of $f(x)$ over the entire closed interval $[a, b]$.

To establish this claim, we choose any $x \in [a, b]$ where $x \neq x_0$. Then there must exist some value j such that $x \in I_j$ but $x \notin I_{j+1}$. Then by the property already noted, $x \in [a, b] - I_{j+1}$ and for any x in this set we know $f(x) \leq f\left(x_{j+1}^*\right) \leq \lim_{j\to\infty} f\left(x_j^*\right) = f\left(\lim_{j\to\infty} x_j^*\right) = f(x_0)$ where in the first equality we apply the Continuous Function Sequence Limit Theorem 10..
We have shown for all $x \in [a, b]$ that $f(x) \leq f(x_0)$. Let $d = x_0$ and the point $(d, f(d))$ is the desired absolute maximum.
QED

28. **Theorem. (Intermediate Value Theorem)**

Let $f(x)$ be continuous on a closed interval $[\,a,b\,]$ where $f(a) \neq f(b)$. Let y be such that $m < y < M$ where m and M are the absolute minimum and maximum of $f(x)$ over $[\,a,b\,]$. Then there exists at least one number c such that $a < c < b$ and $f(c) = y$.

Proof: We will create a sequence of nested closed intervals based on the following defining rules.

First, there exist x-values x_{L1} and x_{R1} such that $m = f(x_{L1})$ and $M = f(x_{R1})$. Without loss of generality we can assume $x_{L1} < x_{R1}$ and we define our first closed interval $I_1 = [\,x_{L1}, x_{R1}\,]$. Note that y lies between $f(x_{L1})$ and $f(x_{R1})$.

Having defined I_j, we recursively define I_{j+1} to be either the left or right half of I_j by a decision rule based on the function value of the midpoint of I_j. If $x_m = \dfrac{x_{Lj} + x_{Rj}}{2}$ denotes the midpoint of I_j then there are two cases to be considered.

If $f(x_m) = y$ then we are done with the entire proof. Note in this case that the midpoint can never be either of the endpoints a or b. Thus when we take $c = x_m$ we know $a < c < b$.

If $f(x_m) \neq y$ then y must lie between either $f(x_{Lj})$ and $f(x_m)$ or else y must lie between $f(x_m)$ and $f(x_{Rj})$. In either case, we define I_{j+1} to be the interval whose endpoints are such that y lies strictly between the function values when the function is evaluated at those endpoints. We rename the new endpoints of I_{j+1} so that $I_{j+1} = [\,x_{Lj+1}, x_{Rj+1}\,]$. Then y is a value that lies between $f(x_{Lj+1})$ and $f(x_{Rj+1})$. Moreover, $I_{j+1} \subset I_j$.

The I_j intervals shrink in length by a factor of $\frac{1}{2}$ at each stage. By the Nested Interval Property 23. of the real numbers we know there exists an x-value that is common to all the closed intervals. Let $x_0 = \bigcap_{j=1}^{\infty} I_j$. Claim that $y = f(x_0)$.

To establish this claim we note that $x_0 = \lim\limits_{j\to\infty} x_{Lj}$ and $x_0 = \lim\limits_{j\to\infty} x_{Rj}$.

$f(x)$ is continuous so by the Continuous Function Sequence Limit Theorem 10.,

$$f(x_0) = f\left(\lim_{j\to\infty} x_{Lj}\right) = \lim_{j\to\infty} f(x_{Lj}) \text{ and } f(x_0) = f\left(\lim_{j\to\infty} x_{Rj}\right) = \lim_{j\to\infty} f(x_{Rj}).$$

So we have shown $\lim\limits_{j\to\infty} f(x_{Lj}) = \lim\limits_{j\to\infty} f(x_{Rj})$.

Now for all j, y lies between $f(x_{Lj})$ and $f(x_{Rj})$ and so by the Squeeze Theorem 3. applied to sequences, since $\lim\limits_{j\to\infty} f(x_{Lj}) = \lim\limits_{j\to\infty} f(x_{Rj})$, $\lim\limits_{j\to\infty} y$ shares the same value as these two limits.

But $y = \lim\limits_{j\to\infty} y$. Thus we have established that $y = f(x_0)$.

QED

29. Theorem. (Continuous Positive Value)

If $f(x)$ is continuous at $x = a$ and if $f(a) > 0$ then there exists an open interval I containing a such that for all $x \in I$, $f(x) > 0$.

Proof: Let $\epsilon = \dfrac{f(a)}{2}$. Then $\epsilon > 0$ since $f(a) > 0$. Since $f(x)$ is continuous at $x = a$, $\lim\limits_{x \to a} f(x) = f(a)$. Thus there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$. Let $I = (a - \delta, a + \delta)$ and let $x \in I$. If $x = a$ then clearly $f(x) > 0$ because $f(a) > 0$. So we may assume $0 < |x - a| < \delta$ and then $|f(x) - f(a)| < \epsilon$. This means $-(f(x) - f(a)) < \epsilon < 2\epsilon = f(a)$. So this means $-f(x) + f(a) < f(a)$. $-f(x) < 0$. $f(x) > 0$. A simpler alternative proof is just to apply the Positive Limit Theorem 5..

QED

30. Corollary. (Continuous Negative Value)

If $f(x)$ is continuous at $x = a$ and if $f(a) < 0$ then there exists an open interval I containing a such that for all $x \in I$, $f(x) < 0$.

Proof: Consider the function $g(x) = -f(x)$. If $f(x)$ is continuous so is $g(x)$ and $g(a) = -f(a) > 0$. So we may apply the Positive Value Theorem 29. to $g(x)$ to conclude $g(x)$ is strictly positive in some neighborhood of a. Then $f(x) = -g(x)$ must be strictly negative in that same neighborhood. A simpler alternative proof is just to apply the Negative Limit Theorem 6..

QED

31. Theorem. (Fermat)

If $f(x)$ has a local extrema at $x = c$ and if $f'(c)$ exists then $f'(c) = 0$.

Proof: First we assume $(c, f(c))$ is a local minimum. Then whenever x is near c, we must have $f(c) \le f(x)$. If we let h denote a sufficiently small positive number then $f(c) \le f(c + h)$. Therefore $0 \le f(c + h) - f(c)$ which also implies $0 \le \dfrac{f(c+h) - f(c)}{h}$. From this last inequality we can apply Theorem 2. and its following Remark 4 to conclude that

$$0 = \lim_{h \to 0^+} 0 \le \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} = f'(c).$$

So far we have only shown that $f'(c) \ge 0$.

Next, if h is a sufficiently small negative number then we still have $f(c) \le f(c + h)$ so that as above we still have $0 \le f(c + h) - f(c)$. But since h is negative, when we divide both sides of this last inequality by h we have $\dfrac{f(c+h) - f(c)}{h} \le 0$. From this last inequality we have

$$f'(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \le \lim_{h \to 0^-} 0 = 0.$$

So $f'(c) \le 0$. We have shown both $f'(c) \ge 0$ and $f'(c) \le 0$. These two statements imply $f'(c) = 0$.

The case in which $(c, f(c))$ is a local maximum is exactly as above with all inequalities reversed.

QED

32. Corollary.

If $f(x)$ has a local extrema at $x = c$, then c is a critical number for the function $f'(x)$.

Proof: If $f'(c)$ is undefined then obviously c is a critical number for $f'(x)$. On the other hand, if $f'(c)$ exists then we can apply Fermat's Theorem 31. to conclude $f'(c) = 0$ which also shows that c is a critical number. In either case c is a critical number.

QED

33. Corollary.

If $f'(c)$ exists and $f'(c) \neq 0$ then $(c, f(c))$ is not a local extrema point.

Proof: This statement is the contrapositive of Corollary 32..

QED

34. Example. A Continuous Function Without Any Extrema

$f(x) = x^3$ is continuous but it has neither a local maximum nor a local minimum and it does not have any absolute extrema either!

35. Theorem.

If $f(x)$ is continuous on $[a, b]$ and $f(x)$ has an absolute extrema at $x = c$ in the open interval (a, b) then c is a critical number for $f'(x)$.

Proof: The point $(c, f(c))$ is a local extrema so apply Corollary 32..

QED

36. Theorem.

Let $f(x)$ be continuous on $[a, b]$. Then $f(x)$ may have local extrema at the endpoints where neither endpoint is a critical number for $f(x)$. $f(x)$ must have absolute extrema at one or both endpoints whenever $f(x)$ does not have any absolute extrema in the open interval (a, b).

Proof: By the Extreme Value Theorem 27. for continuous functions, $f(x)$ has an absolute maximum and an absolute minimum in $[a, b]$. The endpoints can also just be local extrema when f has its absolute extrema at points in the open interval (a, b).

QED

37. Theorem. (Differentiable Implies Continuous)

If $f(x)$ is differentiable at $x = a$ then $f(x)$ is continuous at $x = a$.

Proof: $0 = 0 \cdot f'(a) = \lim\limits_{x \to a} (x - a) \cdot f'(a) = \lim\limits_{x \to a} (x - a) \cdot \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{(x - a)} =$

$$\lim_{x \to a} \left\{ (x - a) \cdot \frac{f(x) - f(a)}{(x - a)} \right\} = \lim_{x \to a} \{f(x) - f(a)\} = \lim_{x \to a} \{f(x)\} - f(a).$$

QED

38. Theorem. (Rolle)

Let $f(x)$ be continuous on the closed interval $[\,a, b\,]$ and differentiable on the open interval (a, b). If $f(a) = f(b)$ then there exists at least one point c, where $a < c < b$ such that $f'(c) = 0$.

Proof: Case 1: $f(x) = k$ for some constant k.

Then c may be chosen as any value between a and b, say let $c = \dfrac{a+b}{2}$.

Case 2: There exists an x in (a, b) such that $f(x) > f(a)$.

Then $f(x)$ has an absolute maximum at some point $x = c$ where $c \neq a$ and $c \neq b$. So c must be in (a, b). Since $f(x)$ is differentiable over (a, b), $f'(c)$ exists. By Fermat's Theorem 31., $f'(c) = 0$.

Case 3: There exists an x in (a, b) such that $f(x) < a$.

Then $f(x)$ has an absolute minimum at some point $x = c$ where $c \neq a$ and $c \neq b$. So c must be in (a, b). Since $f(x)$ is differentiable over $(a, b))$, $f'(c)$ exists. By Fermat's Theorem 31., $f'(c) = 0$.

QED

39. Theorem. (Mean Value Theorem for Derivatives)

Let $f(x)$ be continuous on the closed interval $[\,a, b\,]$ and differentiable on the open interval (a, b).

Then there exists at least one point c, where $a < c < b$ such that $\dfrac{f(b) - f(a)}{b - a} = f'(c)$.

Proof: Compute the equation of the linear function $g(x)$ that is the secant line between $(a, f(a))$ and $(b, f(b))$. Let $g(x) = \dfrac{f(b) - f(a)}{b - a}(x - a) + f(a)$.

Since $g(x)$ is a linear function $g(x)$ is both continuous and differentiable everywhere. In particular $g(x)$ is continuous on $[\,a, b\,]$ and is differentiable over (a, b).

Next, consider the function $h(x)$ that is the difference between $f(x)$ and $g(x)$. $h(x) = f(x) - g(x)$. $h(x)$ is continuous on $[\,a, b\,]$ and is differentiable on (a, b) since both $f(x)$ and $g(x)$ are continuous and differentiable over the respective intervals. Moreover, $h(a) = f(a) - g(a) = f(a) - f(a) = 0$.

Also, $h(b) = f(b) - g(b) = f(b) - f(b) = 0$.

We have shown that $h(a) = h(b)$ so we may apply Rolle's Theorem 38. to the function $h(x)$. We conclude there exists a number c in (a, b) such that $h'(c) = 0$.

But $h'(c) = f'(c) - g'(c)$. This means $f'(c) = g'(c)$ and since $g'(c) = \dfrac{f(b) - f(a)}{b - a}$ we are done because then $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

QED

40. **Theorem. (Increasing/Decreasing Test)**

Let $f(x)$ be such that $f'(x)$ exists over an interval I.
1) If $f'(x) > 0$ on I then $f(x)$ is increasing over I.
2) If $f'(x) < 0$ on I then $f(x)$ is decreasing over I.

Proof: Let x_1 and x_2 be any two numbers in the interval I where $x_1 < x_2$.
By the Mean Value Theorem 39. there exists a number c between x_1 and x_2 such that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. Since $(x_2 - x_1) > 0$ the sign of $f'(c)$ matches the sign of $f(x_2) - f(x_1)$.
So if $f'(c) > 0$ then $f(x_2) > f(x_1)$ meaning $f(x)$ is increasing.
If $f'(c) < 0$ then $f(x_2) < f(x_1)$ meaning $f(x)$ is decreasing.
QED

41. **Theorem. (First Derivative Test)**

Let $f(x)$ be continuous in an open interval containing c where either $f'(c)$ does not exist or $f'(c) = 0$.
1) If $f'(x)$ changes from positive to negative across c then $(c, f(c))$ is a local maximum.
2) If $f'(x)$ changes from negative to positive across c then $(c, f(c))$ is a local minimum.
3) If $f'(x)$ does not change signs across c then $(c, f(c))$ is not a local extrema.

Proof: Apply the Increasing/Decreasing Test 40. to both sides of c.
1) $f(x)$ is increasing on the left and decreasing on the right making a local maximum at $x = c$.
2) $f(x)$ is decreasing on the left and increasing on the right making a local minimum at $x = c$.
3) $f(x)$ is always increasing or always decreasing around $x = c$. This means $x = c$ is neither a local maximum nor a local minimum.

QED

42. **Theorem. (Same Derivatives)**

Let $f(x)$ and $g(x)$ be two functions such that $f'(x) = g'(x)$ for all $x \in (a, b)$.
Then there exists a constant k such that for all $x \in (a, b)$ we have $f(x) = g(x) + k$.

Proof: Consider the function $h(x) = f(x) - g(x)$. For all $x \in (a, b)$ we know $h'(x) = 0$.
So if x_1 and x_2 are both in (a, b) where without loss of generality we assume $x_1 < x_2$, then by the Mean Value Theorem 39. there exists some x such that $x_1 < x < x_2$ and
$\dfrac{h(x_2) - h(x_1)}{x_2 - x_1} = h'(x)$. Since $h'(x) = 0$ for all x, the fraction numerator $h(x_2) - h(x_1) = 0$.
This means $h(x_2) = h(x_1)$ for all x_1 and x_2 in (a, b). So $h(x)$ must be constant over (a, b). If for all $x \in (a, b)$, $h(x) = k$ then $f(x) - g(x) = k$, or $f(x) = g(x) + k$.
QED

43. Definition. Concave Up Function

A function $f(x)$ is called concave up over an interval I if $f(x)$ lies above all of its tangents over I.

44. Definition. Concave Down Function

A function $f(x)$ is called concave down over an interval I if $f(x)$ lies below all of its tangents over I.

45. Theorem. (Concavity Test)

1) If $f''(x) > 0$ for all x in an interval I then $f(x)$ is concave up on I.
2) If $f''(x) < 0$ for all x in an interval I then $f(x)$ is concave down on I.

Proof: 1) Assume $f''(x) > 0$ and let a be any number in I. Must show the graph of $f(x)$ lies above the tangent line given by: $y = f'(a)(x-a) + f(a)$.

So let $x_1 \in I$ where $x_1 \neq a$. We must show $f(x_1) > f'(a)(x_1 - a) + f(a)$.

Case 1: $x_1 > a$.

Apply the Mean Value Theorem 39. to $f(x)$ over $[a, x_1]$. There exists a number c such that $a < c < x_1$ such that $f(x_1) - f(a) = f'(c)(x_1 - a)$.
By the Increasing/Decreasing Theorem 40. we conclude that f' is increasing on I because $f''(x) > 0$ on I. So $f'(a) < f'(c)$ so multiplying this inequality by the positive number $(x_1 - a)$ and then adding $f(a)$ we get $f'(a)(x_1 - a) + f(a) < f'(c)(x_1 - a) + f(a)$.
Finally we substitute $f(x_1) - f(a) = f'(c)(x_1 - a)$ to arrive at
$f'(a)(x_1 - a) + f(a) < f(x_1) - f(a) + f(a)$.
This means $f(x_1) > f'(a)(x_1 - a) + f(a)$ which is all we need show.

Case 2: $x_1 < a$.

This case is similar to Case 1, except we apply the Mean Value Theorem 39. to $f(x)$ over the interval $[x_1, a]$ and we multiply by the negative number $(a - x_1)$.

2) This is similar to the proof of 1) except the inequalities change sense.

QED

46. Theorem. (Second Derivative Test)

Assume $f''(x)$ is continuous near c and assume $f'(c) = 0$.
1) If $f''(c) > 0$ then $(c, f(c))$ is a local minimum.
2) If $f''(c) < 0$ then $(c, f(c))$ is a local maximum.

Proof: 1) By the continuity of $f''(x)$ near c and the fact that $f''(c) > 0$ we know by the Positive Value Theorem 29. $f''(x) > 0$ in some interval containing c. Now apply the above Concavity Test Theorem 45. to conclude $f(x)$ is concave up around c. The point $(c, f(c))$ must be a local minimum.

2) The proof of this part is similar to that for 1) except the graph is concave down and we apply the Negative Value Theorem 30. making the point a local maximum.

QED

47. Definition. Partition of a Closed Interval

Let $[a, b]$ be any closed interval. By a partition of $[a, b]$ we mean any set of $n+1$ points in the interval $[a, b]$, say $\mathbb{P} = \{x_0, x_1, x_2, \ldots, x_n\}$, where $x_0 = a$ and $x_n = b$ and where for $0 \le i \le n-1$ we have $x_i < x_{i+1}$.

48. Definition. A Set of Sample Points Associated With A Partition

Let $\mathbb{P} = \{x_0, x_1, x_2, \ldots, x_n\}$ be any partition of the closed interval $[a, b]$. By a set of sample points associated with $\mathbb{P}$ we mean a set $\{c_1, c_2, c_3, \ldots, c_n\}$ of n points in $[a, b]$ where for $1 \le i \le n$ we have $x_{i-1} \le c_i \le x_i$.

49. Definition. Riemann Sum For A Partition and Sample Point Set

Let $f(x)$ be a continuous function over a closed interval $[a, b]$. Let $\mathbb{P} = \{x_0, x_1, x_2, \ldots, x_n\}$ be any partition of $[a, b]$ and let $\{c_1, c_2, c_3, \ldots, c_n\}$ be a set of sample points associated with $\mathbb{P}$. Then $\sum\limits_{i=1}^{n} f(c_i)(x_i - x_{i-1})$ is called the Riemann sum associated with the partition $\mathbb{P}$ and the set of sample points c_i.

50. Definition. Norm Of A Partition

Let $\mathbb{P} = \{x_0, x_1, x_2, \ldots, x_n\}$ be any partition of the closed interval $[a, b]$. We denote the norm or the mesh of $\mathbb{P}$ by $\|\mathbb{P}\|$ and define this to be the number that is the maximum of the numbers $\Delta x_i = (x_i - x_{i-1})$.

51. Definition. Definite Integral of a Continuous Function

Let $f(x)$ be a continuous function over a closed interval $[a, b]$. Let $\|\mathbb{P}\|$ denote the norm of any partition $\mathbb{P}$ of the closed interval $[a, b]$. Let $\{c_1, c_2, c_3, \ldots, c_n\}$ be any set of sample points associated with $\mathbb{P}$.

$$\int_a^b f(x)dx = \lim_{\|\mathbb{P}\| \to 0} \left(\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) \right)$$

provided this limit exists.

52. Definition. Signed Area For A Function

Let $f(x)$ be a continuous function over a closed interval $[a, b]$. We define the ***signed area*** associated with the function $f(x)$ and the closed interval $[a, b]$ to be the number

$$\int_a^b f(x)dx$$

53. Definition. Average Value For A Function

Let $f(x)$ be a continuous function over a closed interval $[\,a, b\,]$. We define the average value associated with the function $f(x)$ and the closed interval $[\,a, b\,]$ to be the number

$$\frac{1}{b-a}\int_a^b f(x)dx$$

54. Theorem. (Comparison of Integrals)

Let $f(x)$ and $g(x)$ both be continuous over $[\,a, b\,]$ and assume for all $x \in [\,a, b\,]$ we have $f(x) \leq g(x)$.

$$\text{Then } \int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

Proof: Let $\mathbb{P} = \{x_0, x_1, x_2, \ldots, x_n\}$ be any partition of the closed interval $[\,a, b\,]$ and let $\{c_1, c_2, c_3, \ldots, c_n\}$ be a set of sample points associated with the partition $\mathbb{P}$. Then for $1 \leq i \leq n$ we know $f(c_i) \leq g(c_i)$. Thus,

$$\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) \leq \sum_{i=1}^{n} g(c_i)(x_i - x_{i-1}).$$

Note that the above inequality is independent of the size of $\|\mathbb{P}\|$. Thus we conclude by a process similar to the Limit Inequality Theorem 2. that

$$\lim_{\|\mathbb{P}\|\to 0}\left(\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})\right) \leq \lim_{\|\mathbb{P}\|\to 0}\left(\sum_{i=1}^{n} g(c_i)(x_i - x_{i-1})\right)$$

and this means that $\displaystyle\int_a^b f(x)dx \leq \int_a^b g(x)dx.$

QED

55. Theorem. (Mean Value Theorem For Integrals)

Let $f(x)$ be a continuous function over a closed interval $[a, b]$. Then there exists at least one number c such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

Proof: By the Extreme Value Theorem 27. for continuous functions, there exist numbers m and M where m is an absolute minimum of $f(x)$ and M is an absolute maximum of $f(x)$ over $[a, b]$ and there exist values p and q such that

1) $f(p) = m$
2) $f(q) = M$
3) $p \in [a, b]$ and $q \in [a, b]$.
4) For all $x \in [a, b]$, $m \leq f(x) \leq M$.

Now by the Comparison Test For Integrals 54.,

$$\int_a^b m \cdot dx \leq \int_a^b f(x)dx \leq \int_a^b M \cdot dx$$

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

$$f(p) = m \leq \frac{1}{b-a}\int_a^b f(x)dx \leq M = f(q)$$

We have shown $\frac{1}{b-a}\int_a^b f(x)dx$ is a value between the two y-coordinates $f(p)$ and $f(q)$. Since $f(x)$ is continuous over $[a, b]$ we may apply the Intermediate Value Theorem 28. for continuous functions to conclude there exists a number c strictly between a and b such that $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$.

QED

56. **Theorem. (Fundamental Theorem of Calculus)**

Let $f(t)$ be a continuous function over a closed interval $[\,a, b\,]$.

1) The function $F(x) = \displaystyle\int_a^x f(t)dt$ has a derivative at every point $x \in [\,a, b\,]$ and moreover $\dfrac{dF}{dx} = f(x)$.

2) $\displaystyle\int_a^b f(x)dx = G(b) - G(a)$ where $G(x)$ is any antiderivative of $f(x)$.

Proof: 1) $$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{\displaystyle\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h} =$$

$$\lim_{h \to 0} \frac{\displaystyle\int_x^{x+h} f(t)dt}{h} = \lim_{h \to 0} \frac{f(c) \cdot h}{h} = \lim_{h \to 0} f(c)$$ where c depends on h but c

is such that $x < c < x + h$ when $h > 0$ and c is such that $x + h < c < x$ when $h < 0$. The second to last equality above makes use of the Mean Value Theorem for Integrals 55.. In either case, as $h \to 0,\ c \to x$ so that $\displaystyle\lim_{h \to 0} f(c) = f(x)$.

So continuing to take this limit, $F'(x) = \displaystyle\lim_{h \to 0} f(c) = f(x)$.

2) Let $G(x)$ be any antiderivative of $f(x)$ and let $F(x)$ be defined as in part 1). Then both $G(x)$ and $F(x)$ are anti-derivatives of the same $f(x)$ function so by Theorem 42. there must exist a constant k such that $G(x) = F(x) + k$. Finally,

$$G(b) - G(a) = \{F(b) + k\} - \{F(a) + k\} = F(b) - F(a) =$$

$$\int_a^b f(t)dt - \int_a^a f(t)dt = \int_a^b f(t)dt - 0 = \int_a^b f(t)dt.$$

QED

57. Example: A Continuous Function $f(x)$ where $f'(x)$ Is Not Continuous

Let $f(x) = \begin{cases} x^2 \cdot sin\left(\frac{1}{x}\right) & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$ Then,

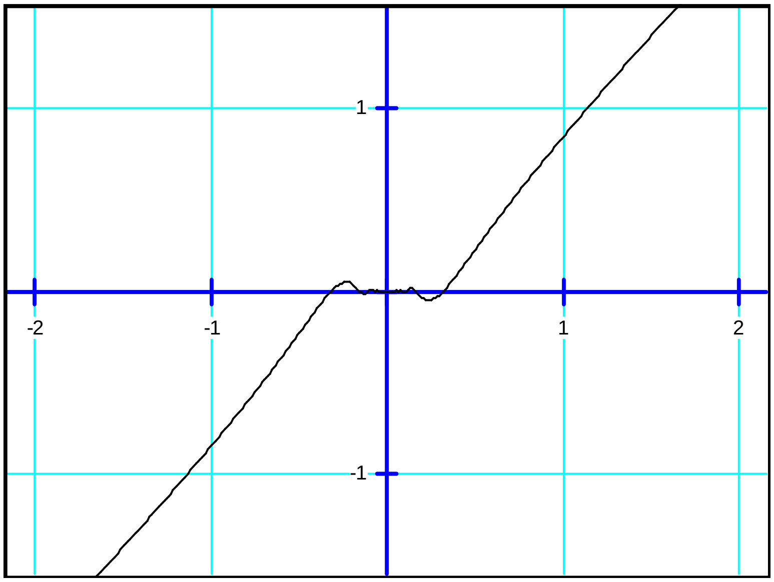
1) $f'(0) = 0$
2) if $x \neq 0$ then $f'(x) = 2xsin\left(\frac{1}{x}\right) - cos\left(\frac{1}{x}\right)$
3) $f'(x)$ is not continuous at $x = 0$ because $\lim\limits_{x \to 0} f'(x) \neq f'(0)$.

Proof: 1) $f'(0) = \lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{f(h) - 0}{h} = \lim\limits_{h \to 0} \dfrac{f(h)}{h} =$
$\lim\limits_{h \to 0} \dfrac{h^2 sin\left(\frac{1}{h}\right)}{h} = \lim\limits_{h \to 0} h \cdot sin\left(\frac{1}{h}\right)$. Now $-1 \leq sin\left(\frac{1}{h}\right) \leq +1$ so if $h > 0$ then $-h \leq h \cdot sin\left(\frac{1}{h}\right) \leq h$ so applying the Squeeze Theorem 3. we have $\lim\limits_{h \to 0^+} h \cdot sin\left(\frac{1}{h}\right) = 0$. If $h < 0$ then we can again multiply by h, but then we must reverse the inequality to get $h > h \cdot sin\left(\frac{1}{h}\right) > -h$. Again by the Squeeze Theorem 3. $\lim\limits_{h \to 0^-} h \cdot sin\left(\frac{1}{h}\right) = 0$. Since both one-sided limits make 0, we have shown $f'(0) = 0$.

2) If $x \neq 0$, we can apply the product rule to compute
$f'(x) = 2x \cdot sin\left(\frac{1}{x}\right) + x^2 \cdot cos\left(\frac{1}{x}\right) \cdot (-1)x^{-2} = 2xsin\left(\frac{1}{x}\right) - cos\left(\frac{1}{x}\right)$

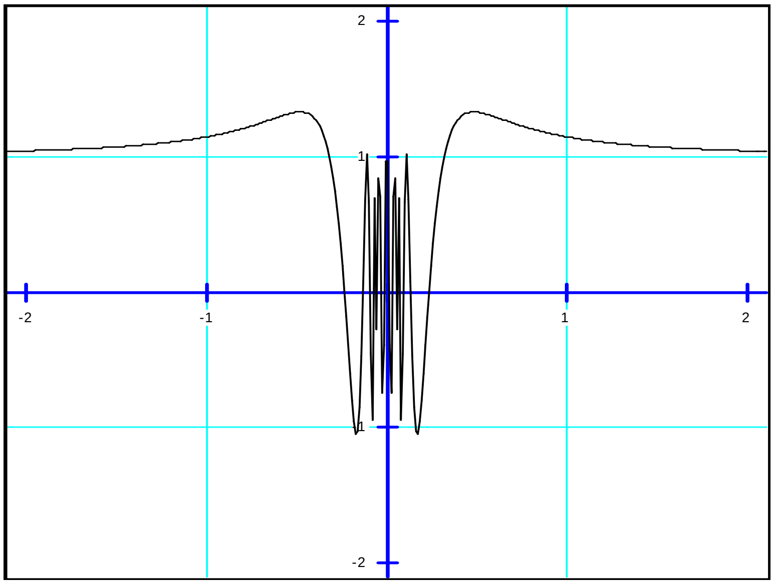
3) $\lim\limits_{x \to 0} \left\{2xsin\left(\frac{1}{x}\right) - cos\left(\frac{1}{x}\right)\right\}$ D.N.E. because the first term approaches 0 while $\lim\limits_{x \to 0} cos\left(\frac{1}{x}\right)$ D.N.E..

The figure below shows the graph of $y = f(x)$ which has small damped oscillations near the origin. See also, the next figure which shows the graph of $y = f'(x)$.



Example continued on the next page.

The figure below shows the graph of $y = f'(x)$ whose oscillations near the origin are not damped at all; the oscillations really vary between -1 and $+1$.



58. Theorem. (Intermediate Values For Derivatives)

Let $f(x)$ be differentiable over a closed interval $[\,a, b\,]$. If $f'(a) = A$ and $f'(b) = B$ and if C is between A and B then there exists at least one value $c \in (a, b)$ such that $f'(c) = C$.

Proof: Without loss of generality we assume $A < C < B$.

From the limit definition for any derivative, we can find a positive number h such that

$$\frac{f(a+h) - f(a)}{h} < C \text{ and } C < \frac{f(b-h) - f(b)}{-\,h}$$

So the function $F(x) = \dfrac{f(x+h) - f(x)}{h}$ has the property that $F(a) < C < F(b-h)$.

So C is an intermediate value for the function $F(x)$. Now since $f(x)$ is differentiable over the closed interval $[\,a, b\,]$, it is also continuous over that same interval. So $F(x)$ is continuous over $[\,a, b\,]$ and by the Intermediate Value Theorem 28. we know there is a point x_0 such that $F(x_0) = C$ where x_0 is such that $a < x_0 < b - h$.

By the Mean Value Theorem 39. applied to $f(x)$ over the interval $[x_0,\ x_0 + h]$ there exists a point c such that

$$f'(c) = \frac{f(x_0 + h) - f(x_0)}{h} = F(x_0) = C$$

where $x_0 < c < x_0 + h$. Now from the first inequality above for x_0 and this last inequality for c we have $a < x_0 < c < x_0 + h < b$. In other words, $c \in (a, b)$.

QED

59. Theorem. (Linear Approximation)

Let $f(x)$ be differentiable at $x = a$. Then the function $L(x) = f(a) + f'(a)(x - a)$ is called the Linear Approximation to $f(x)$ at $x = a$ and has the property that $L(a) = f(a)$ and $L'(a) = f'(a)$ and moreover $\lim\limits_{x \to a} L(x) = \lim\limits_{x \to a} f(x)$. This means when x is close to a, $f(x) \approx f(a) + f'(a)(x - a)$.

Proof: It is trivial that $L(a) = f(a)$ and that $L'(a) = f'(a)$.

The more significant property is that $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} L(x)$. To prove this limit we only need to compute $\lim\limits_{x \to a} L(x) = \lim\limits_{x \to a} f(a) + f'(a)(x - a) = f(a)$. Now since $f(x)$ is assumed differentiable at $x = a$, by Theorem 37., $f(x)$ is continuous at $x = a$ and by continuity we know $f(a) = \lim\limits_{x \to a} f(x)$. Each of the two limits, $\lim\limits_{x \to a} L(x)$ and $\lim\limits_{x \to a} f(x)$, have been shown to be the same as $f(a)$ so both limits must equal each other.

QED

60. Theorem. (nth Degree Taylor Polynomial Approximation)

Let $f(x)$ be differentiable n times at $x = a$. Then the function $p(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x - a)^i =$
$f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$ is called the nth Degree Taylor Polynomial Approximation to $f(x)$ at $x = a$. $p(x)$ is a polynomial of degree n.

It has the property that for $i = 0, 1, 2, 3, \ldots, n$ $p^{(i)}(a) = f^{(i)}(a)$. Moreover, $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} p(a)$.
When x is close to a we have:

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

Proof: Clearly $p(a) = f(a)$ and by continually differentiating $p(x)$ and then substituting $x = a$ we find that when $i \geq 1$, $p^{(i)}(a) = f^{(i)}(a)$. That $\lim\limits_{x \to a} p(x) = f(a)$ is just as trivial. Since $f(x)$ is differentiable at $x = a$, by Theorem 37., $f(x)$ is continuous at $x = a$ so we also have $\lim\limits_{x \to a} f(x) = f(a) = \lim\limits_{x \to a} p(a)$.

QED

61. Theorem. (Even Higher Derivative Test)

Assume $f(x)$ is differentiable in a neighborhood of $x = a$ and assume

$f'(a) = f''(a) = f'''(a) = \cdots = f^{(n-1)}(a) = 0$ and $f^{(n)}(a) \neq 0$ where $n \geq 2$.

1) If n is even and $f^{(n)}(a) < 0$ then $f(a)$ is a local maximum.

2) If n is even and $f^{(n)}(a) > 0$ then $f(a)$ is a local minimum.

3) If n is odd then $f(a)$ is neither a local maximum nor a local minimum.

Proof: 1) Consider the nth Degree Taylor Polynomial Approximation to the function $f(x)$ as given in Theorem 60.. When x is very close to a we have the approximation:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots +$$

$$+ \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{(n-1)} + \frac{f^{(n)}(a)(x-a)^n}{n!}.$$

If the first $(n-1)$ derivatives of $f(x)$ vanish at $x = a$ then this approximation simplifies to just the sum of two terms.

$$f(x) \approx f(a) + \frac{f^{(n)}(a)(x-a)^n}{n!}$$

Note that if n is even then $(x-a)^n$ is positive so the sign of the 2nd term is determined entirely by the sign of $f^{(n)}(a)$. So if $f^{(n)}(a) < 0$ and if x is chosen sufficiently close to a it is clear that $f(x) \approx f(a) + (a\ negative\ number)$. This implies $f(x) \leq f(a)$ when x is near a. Thus in this case $f(a)$ is a local maximum.

2) This part is identical to part 1) except now the second term is positive.
$f(x) \approx f(a) + (a\ positive\ number)$. This implies $f(x) \geq f(a)$ when x is near a.
Thus in this case $f(a)$ is a local minimum.

3) Again, refer to part 1).

$$f(x) \approx f(a) + \frac{f^{(n)}(a)(x-a)^n}{n!}$$

Now when n is odd, the sign of $f^{(n)}(a)$ is irrelevant to the sign of the second term because the second term changes sign depending on whether $x > a$ or $x < a$. Since x can be near a but on either side of a, we get both:
$f(x) \approx f(a) + (a\ negative\ number)$ and $f(x) \approx f(a) + (a\ positive\ number)$.
Clearly the only conclusion that can be drawn is that $f(a)$ is neither a local maximum nor a local minimum in this part.

QED

62. Remark: The above Theorem 61. is a generalization of the Second Derivative Test 46..

63. Theorem. (Generalized Mean Value Theorem; Cauchy)

Assume $f(x)$ and $g(x)$ are both continuous on $[a, b]$ and that both functions are differentiable on (a, b) and further assume that for all $x \in (a, b)$, $g'(x) \neq 0$. Then there exists at least one number c where $a < c < b$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof: Let $h(x) = [g(b) - g(a)] \cdot [f(x) - f(a)] - [g(x) - g(a)] \cdot [f(b) - f(a)]$.
Then it is trivial to show that $h(a) = 0 = h(b)$. It is also easy to show that the remaining hypotheses of Rolle's Theorem 38. are satisfied by the $h(x)$ function. So we can find at least one number c such that $h'(c) = 0$. But $h'(x) = [g(b) - g(a)] \cdot f'(x) - g'(x) \cdot [f(b) - f(a)]$.
Now we can substitute $x = c$ and arrive at the equation

$$[g(b) - g(a)] \cdot f'(c) = g'(c) \cdot [f(b) - f(a)]$$

from which the equation in the theorem statement immediately follows if we divide both sides of this equation by the term $g'(c) \cdot [g(b) - g(a)]$.

Note that we use the fact that $g'(c) \neq 0$ to establish the first factor is nonzero. We also need to apply the Mean Value Theorem 39. to $g(x)$ to be able to write $g(b) - g(a) = (b - a) \cdot g'(c)$ and then we can use this fact to conclude that the other factor $[g(b) - g(a)] \neq 0$. So dividing both sides of the above equation by the nonzero term $g'(c) \cdot [g(b) - g(a)]$ is what yields the equation in the theorem statement.

QED

64. Remark.

The statement of the Generalized Mean Value Theorem 63. reduces to that of the Mean Value Theorem 39. if the function $g(x) = x$.

Index