

# **The Pure Cubic Polynomial and Intersections With Linear Functions**

**by**

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# The Pure Cubic Polynomial and Intersections With Linear Functions

**Theorem 1.** If  $m \leq 0$  then the line  $y = mx + b$  intersects the graph of  $y = x^3$  in exactly one point.

Proof:

Consider evaluating the polynomial function  $y = x^3 - mx - b$ , first when  $x$  is very large and positive, and then when  $x$  is very large and negative. We would find two  $y$  values with opposite signs. By the continuity of polynomials we could apply the Intermediate Value Theorem to guarantee the existence of a solution to the equation  $x^3 - mx - b = 0$ . Call the value of  $x$  that makes the zero  $r_1$ .

Consider evaluating the polynomial  $x^3 - mx - b$  at any root  $x = r_1$  using synthetic substitution.

$$\begin{array}{r|rrrr}
 & & & & \boxed{r_1} \\
 & 1 & 0 & -m & -b \\
 r_1 & & r_1 & (r_1)^2 & (r_1)^3 - m(r_1) \\
 \hline
 & 1 & r_1 & (r_1)^2 - m & (r_1)^3 - m(r_1) - b
 \end{array}$$

The last entry in the last row is really 0, and the first three entries of the last row are the coefficients of the deflated polynomial that is  $x^2 + r_1x + (r_1)^2 - m$ . If we compute the discriminant of this quadratic we see its value is  $(r_1)^2 - 4 \cdot 1 \cdot [(r_1)^2 - m] = -3(r_1)^2 + 4m$ . If either  $r_1 \neq 0$  or  $m$  is negative, this value is negative and that implies there is no other real solution to  $x^3 - mx - b = 0$ .

If  $r_1 = 0$  and  $m = 0$  then  $b$  must be zero as well, but in this case the line degenerates to  $y = 0$ , that is the  $x$ -axis. The only point where the  $x$ -axis intersects the graph of  $y = x^3$  is at the origin. In all cases there is one and only one intersection point.

**Theorem 2.** Assume  $r_1$  and  $r_2$  are two distinct real roots of the real polynomial equation  $x^3 - mx - b = 0$ .

Further assume these are the only two real number roots of this equation. Then,

- 1)  $m$  must be positive.
- 2) One of these roots must be a root of multiplicity two.
- 3) The root with single multiplicity is  $-2$  times the root with multiplicity two.

Proofs:

1). Any solution to the cubic equation  $x^3 - mx - b = 0$  can be considered as an intersection point of the graphs of the two functions  $y = x^3$  and  $y = mx + b$ . That  $m$  must be positive is because if  $m \leq 0$ , then by **Theorem 1** we have a contradiction that  $r_1$  and  $r_2$  are two distinct roots. By **Theorem 1** in the case where  $m \leq 0$  there should be only one intersection point.

2). If  $r_1$  is a root of multiplicity two then we are done. So we will argue that if  $r_1$  has multiplicity one, then  $r_2$  must be a root of multiplicity two. Consider the deflated quadratic as was found in **Theorem 1**.

$$x^2 + r_1x + [(r_1)^2 - m]$$

Since both  $r_1$  and  $r_2$  are roots of the original cubic equation we know

$$(r_1)^3 - mr_1 - b = 0 = (r_2)^3 - mr_2 - b$$

$$(r_1)^3 - (r_2)^3 = mr_1 - mr_2$$

$$(r_1 - r_2)[(r_1)^2 + r_1r_2 + (r_2)^2] = m(r_1 - r_2)$$

Since  $r_1$  is distinct from  $r_2$  we can divide both sides of the above equation by their difference and conclude that

$$(r_1)^2 + r_1r_2 + (r_2)^2 = m$$

Now we will simply re-write the deflated quadratic after substituting this value for  $m$ . The new form of the deflated quadratic is:

$$\begin{aligned} x^2 + r_1x + [(r_1)^2 - m] &= x^2 + r_1x + [(r_1)^2 - (r_1)^2 - r_1r_2 - (r_2)^2] \\ &= x^2 + r_1x + [-r_1r_2 - (r_2)^2] \end{aligned}$$

Now watch what happens when we synthetically substitute  $x = r_2$ . We have:

$$\begin{array}{r|rrr} & & & r_2 \\ 1 & r_1 & -r_1r_2 - (r_2)^2 & \\ & r_2 & r_1r_2 + (r_2)^2 & \\ \hline 1 & r_1 + r_2 & 0 & \end{array}$$

We have shown  $r_2$  is a root of the deflated polynomial. This means  $r_2$  is a root of multiplicity at least one of the original cubic polynomial. From above we see the remainder polynomial is the linear function  $x + r_1 + r_2$ . Surely even this function has a simple zero. That zero cannot be  $r_1$  because we are assuming  $r_1$  has multiplicity one. That zero must be  $r_2$ , because if it is neither  $r_1$  nor  $r_2$  then it must be a third root that contradicts that the original cubic polynomial has only two roots. Thus  $r_2$  must also be the root of  $x + r_1 + r_2 = 0$ . We have established that  $r_2$  is a root of multiplicity two of the original cubic polynomial.

- 3) We can perform synthetic substitution using the remainder polynomial in the last line of the above table. Since  $r_2$  is the root of the linear function  $x + r_1 + r_2$ , when we again perform synthetic substitution

$$\begin{array}{r|rr} & & r_2 \\ 1 & r_1+r_2 & \\ & r_2 & \\ \hline 1 & r_1 + 2r_2 & \end{array}$$

we know the last number in the last row must be 0.  $r_1 + 2r_2 = 0$ . This implies  $r_1 = -2r_2$ . The root with single multiplicity is  $-2$  times the root with double multiplicity.

**Theorem 3.** Assume the line  $y = mx + b$  intersects the curve  $y = x^3$  in exactly two distinct points. Then the following five statements must be true.

- 1) At one of the intersection points, the line  $y = mx + b$  is a tangent line to the curve  $y = x^3$ .
- 2) At the point of tangency, the square of the  $x$ -coordinate is  $\frac{1}{3}m$ .
- 3) At the point of tangency, the  $y$ -coordinate is  $-\frac{1}{2}b$ .
- 4)  $m$  and  $b$  must be related to each other by the equation  $m = 3\left(\sqrt[3]{-\frac{1}{2}b}\right)^2$ .
- 5)  $b \neq 0$  and the point of tangency is never at the origin. The point of tangency is in the first quadrant if  $b < 0$  and it is in the third quadrant if  $b > 0$ . In other words, if  $(x, y)$  is the point of tangency then  $x$  and  $y$  have the same sign and that sign is the opposite of  $b$ 's sign.

**Proof:** Assume the  $x$ -coordinates of the two points of intersection are denoted by  $r_1$  and  $r_2$ . By **Theorem 2** we know one of these, call it  $r_2$ , has multiplicity two, while  $r_1$  will denote the root with multiplicity one. We claim the line  $y = mx + b$  is a tangent line to the curve  $y = x^3$ , at  $x = r_2$ , the double root. We must establish this fact. To begin, we can reason we must have

$$(x - r_1)(x - r_2)^2 = x^3 - mx - b$$

and after expanding the polynomial on the left and equating coefficients with the polynomial on the right we get the following system of three equations:

$$\begin{cases} -r_1 - 2r_2 = 0 \\ 2r_1r_2 + (r_2)^2 = -m \\ -r_1(r_2)^2 = -b \end{cases}$$

The first equation tells us  $r_1 = -2r_2$ , a fact we already know from **Theorem 2**. Equating coefficients is just an alternative way of establishing this fact.

The tangent line slope to  $y = x^3$  would be  $3x^2$  at any point. When  $x = r_2$  the tangent line slope is  $3(r_2)^2$ .

Now we will determine this slope in a different way from the above system of three equations. We will substitute for  $r_1$ , after we solve the second equation for  $m$ .

$$r_1 = -2r_2 \text{ so } m = -2r_1r_2 - (r_2)^2 = -2(-2r_2)r_2 - (r_2)^2 = 4(r_2)^2 - (r_2)^2 = 3(r_2)^2$$

At  $x = r_2$ , the  $y$ -coordinate on  $y = x^3$  is given by  $(r_2)^3$ . At  $x = r_2$  on the line  $y = mx + b$ , the  $y$ -coordinate is  $m(r_2) + b = 3(r_2)^2(r_2) + r_1(r_2)^2 = 3(r_2)^3 + (-2r_2)(r_2)^2 = (r_2)^3$ .

So we have established that both the slope and the point of tangency match. The line  $y = mx + b$  is a tangent line to  $y = x^3$  at the point  $(r_2, (r_2)^3)$ . In fact, we could write the tangent line  $y = mx + b$  in the form

$$y = 3(r_2)^2x + [-2(r_2)^3]$$

Next,  $m = 3(r_2)^2$  or  $(r_2)^2 = \frac{1}{3}m$ . In other words, the square of the  $x$ -coordinate at the point of tangency is one-third of the slope of that tangent line.

We also have  $b = -2(r_2)^3$  so the cube of the  $x$ -coordinate  $r_2$  is  $-\frac{1}{2}b$  and this is also the  $y$ -coordinate at that point of tangency.  $(r_2)^3$  is the  $y$ -coordinate at the point of tangency. Finally, we can directly solve for  $r_2$  in terms of  $b$  and write  $m = 3(r_2)^2 = 3\left(\sqrt[3]{-\frac{1}{2}b}\right)^2$ . Since  $m > 0$  we know  $b \neq 0$ . At the point of tangency,  $y = -\frac{1}{2}b$  and since  $b \neq 0, y \neq 0$  and  $y$  and  $b$  have opposite signs. At the point of tangency  $x^3 = y$  and since  $y \neq 0, x \neq 0$ , and  $x$  has the same sign as  $y$ , the opposite of  $b$ 's sign.

**Theorem 4.** Assume the real polynomial equation  $x^3 - mx - b = 0$  has three distinct real roots. Then  $m$  must be positive. Moreover, any two of these roots are such that their sum is the opposite of the third root.

Proof: Denote the three roots by  $r_1$  and  $r_2$  and  $r_3$ . To establish that  $m$  is positive we could repeat the argument in **Theorem 2**, while using the fact that we have more than one real root. Consider the synthetic substitution of  $r_1$  into the cubic polynomial.

$$\begin{array}{r|rrrr} & & & & r_1 \\ 1 & 0 & -m & & -b \\ & r_1 & (r_1)^2 & (r_1)^3 - m(r_1) & \\ \hline 1 & r_1 & (r_1)^2 - m & (r_1)^3 - m(r_1) - b & \end{array}$$

Since  $r_1$  is a root, the deflated polynomial is the quadratic  $x^2 + r_1x + (r_1)^2 - m$  that is seen in the last line above. The other two roots,  $r_2$  and  $r_3$ , must be the two solutions to this quadratic. Applying the quadratic formula, we can assume without loss of generality that

$$r_2 = \frac{-r_1 + \sqrt{(r_1)^2 - 4[(r_1)^2 - m]}}{2} \quad \text{and} \quad r_3 = \frac{-r_1 - \sqrt{(r_1)^2 - 4[(r_1)^2 - m]}}{2}$$

Now it is trivial that  $r_2 + r_3 = -r_1$  because the radical expressions cancel each other when we add the fractions. Moreover, since we could have labeled the three roots in any order, we have actually proved the full general statement of the theorem.

**Theorem 5.** Every line in the form  $y = mx + b$  must intersect the graph of  $y = x^3$  in at least one point.

Assume  $P(r_1, (r_1)^3)$  is an intersection point and further assume  $m > 0$ .

- 1)  $P$  is the only intersection point if and only if  $4m < 3(r_1)^2$ .
- 2) There are exactly two intersection points if and only if one of the intersection points has an  $x$ -coordinate  $r \neq 0$  such that  $m = 3r^2$ .
- 3) There are three distinct intersection points if and only if there exists at least one point of intersection  $x$ -coordinate  $r$ , such that  $m > 3r^2$ .

Proof: The idea in the first part of this theorem is nearly identical to that in **Theorem 1**. We can guarantee a point of intersection exists by evaluating the function  $y = x^3 - mx - b$  at both very large negative and very large positive values of  $x$ . This will produce two  $y$ -coordinates that differ in sign. We then apply the Intermediate Value Theorem for continuous functions to get a value  $r_1$  such that  $(r_1)^3 = mr_1 + b$ .

We then factor  $x^3 - mx - b = (x - r_1)[x^2 + r_1x + (r_1)^2 - m]$  and consider the nature of the quadratic factor along with the value of  $m$ . The discriminant of the quadratic is:

$$(r_1)^2 - 4 \cdot [(r_1)^2 - m] = -3(r_1)^2 + 4m$$

1) It should be clear that  $P$  is the only intersection point if and only if this discriminant is negative. In other words,  $P$  is the only intersection point if and only if  $4m < 3(r_1)^2$ .

2) If there are exactly two intersection points then both **Theorem 2** and **Theorem 3** can be applied. In the proof of **Theorem 3** we showed that  $m = 3r^2$  where  $r$  denotes the root with multiplicity two. This equation implies  $r \neq 0$  since  $m > 0$ .

Now for the converse of part 2) let's assume  $m = 3r^2$  where  $mr + b = r^3$ . Then since  $m > 0$ ,  $r \neq 0$ . We must show there are two intersection points. In this case we know  $3r^2 \cdot r + b = r^3$ . This means  $b = -2r^3$ . We then write the original cubic polynomial equation as  $x^3 - 3r^2x + 2r^3 = 0$ . Let's synthetically substitute here to find the remainder polynomial when  $x = r$ .

$$\begin{array}{r|rrrr} & & & & r \\ 1 & 0 & -3r^2 & 2r^3 & \\ & r & r^2 & -2r^3 & \\ \hline 1 & r & -2r^2 & 0 & \end{array}$$

We will establish  $r$  is a double root by performing another synthetic substitution using the remainder polynomial we just computed.

$$\begin{array}{r|rrr} & & & r \\ 1 & r & -2r^2 & \\ & r & 2r^2 & \\ \hline 1 & 2r & 0 & \end{array}$$

We have established that  $r$  is the root with double multiplicity. From the last line in this table we know the second distinct root  $x$ -coordinate must be where  $x = -2r$ . We claim the two points of intersection must be  $P(r, r^3)$  and  $Q(-2r, -8r^3)$ . These points are distinct because  $r \neq 0$ . It is clear that both  $P$  and  $Q$  are on  $y = x^3$ . We need only establish that  $Q$  is on the line  $y = mx + b$ . But that line can be written as  $y = 3r^2x - 2r^3$  and when we let  $x = -2r$  we find  $y = 3r^2(-2r) - 2r^3 = -6r^3 - 2r^3 = -8r^3$ . Thus  $Q$  is also on the line. We have explicitly shown the existence of the two intersection points.

3) If there are three distinct roots then we can assume none of these are zero, because if any root  $r = 0$ , then it is trivial to prove  $m > 3r^2$  because  $m > 0 = 3r^2$ . So we need only prove the forward statement for the case where none of the roots is 0.

Without loss of generality we can assume  $r_1$  is the  $x$ -coordinate of the root that is smallest in absolute value. Another way of saying this is that  $r_1$  is the root that is closest to 0 on the number line. We let  $r_2$  and  $r_3$  denote the other two roots.

Subcase 1:  $r_1 > 0$ .

If one of the other two roots is positive, we can assume it is  $r_2$ .

Then since  $r_2 \neq r_1$  and  $r_1$  is closer to 0, we can write  $0 < r_1 < r_2$ . From this inequality we know  $(r_1)^2 < r_1 r_2$  and we know  $(r_1)^2 < (r_2)^2$ .

Now consider that the two points  $P(r_1, (r_1)^3)$  and  $Q(r_2, (r_2)^3)$  must be on the line  $y = mx + b$  whose slope  $m$  can be computed the old-fashioned way as:

$$m = \frac{(r_2)^3 - (r_1)^3}{r_2 - r_1} = \frac{(r_2 - r_1)[(r_2)^2 + r_2 r_1 + (r_1)^2]}{(r_2 - r_1)} = (r_2)^2 + r_1 r_2 + (r_1)^2$$

Now we derive that  $m = (r_2)^2 + r_1 r_2 + (r_1)^2 > (r_1)^2 + (r_1)^2 + (r_1)^2 = 3(r_1)^2$ .

If both of the other two roots are negative then we can assume  $r_3$  is the most negative and we can write  $r_3 < r_2 < -r_1 < 0$ .

We then have  $(r_3)^2 > (r_2)^2$  and  $r_2 r_3 > (r_2)^2$  and  $(r_2)^2 > (r_1)^2$ .

Now consider that the two points  $P(r_2, (r_2)^3)$  and  $Q(r_3, (r_3)^3)$  must be on the line  $y = mx + b$  whose slope  $m$  can be computed as:

$$m = \frac{(r_3)^3 - (r_2)^3}{r_3 - r_2} = \frac{(r_3 - r_2)[(r_3)^2 + r_3 r_2 + (r_2)^2]}{(r_3 - r_2)} = (r_3)^2 + r_3 r_2 + (r_2)^2$$

Now we derive that

$$m = (r_3)^2 + r_2 r_3 + (r_2)^2 > (r_2)^2 + (r_2)^2 + (r_2)^2 = 3(r_2)^2 > 3(r_1)^2$$

Subcase 2:  $r_1 < 0$ .

If one of the other two roots is negative, we can assume it is  $r_2$ .

Then since  $r_2 \neq r_1$  we can write  $r_2 < r_1 < 0$  and then we have  $r_1 r_2 > (r_1)^2$  and we have  $(r_1)^2 < (r_2)^2$ .

Now consider that the two points  $P(r_1, (r_1)^3)$  and  $Q(r_2, (r_2)^3)$  must be on the line  $y = mx + b$  whose slope  $m$  can be computed as:

$$m = \frac{(r_2)^3 - (r_1)^3}{r_2 - r_1} = \frac{(r_2 - r_1)[(r_2)^2 + r_2 r_1 + (r_1)^2]}{(r_2 - r_1)} = (r_2)^2 + r_1 r_2 + (r_1)^2$$

Now we derive that  $m = (r_2)^2 + r_1 r_2 + (r_1)^2 > (r_1)^2 + (r_1)^2 + (r_1)^2 = 3(r_1)^2$ .

If both of the other two roots are positive then we can assume  $r_3$  is the most positive and we can write  $0 < -r_1 < r_2 < r_3$ .

We also have  $(r_3)^2 > (r_2)^2$  and  $r_2 r_3 > (r_2)^2$  and  $(r_2)^2 > (r_1)^2$ .

Now consider that the two points  $P(r_2, (r_2)^3)$  and  $Q(r_3, (r_3)^3)$  must be on the line  $y = mx + b$  whose slope  $m$  can be computed as:

$$m = \frac{(r_3)^3 - (r_2)^3}{r_3 - r_2} = \frac{(r_3 - r_2)[(r_3)^2 + r_3 r_2 + (r_2)^2]}{(r_3 - r_2)} = (r_3)^2 + r_3 r_2 + (r_2)^2$$

Now we derive that

$$m = (r_3)^2 + r_2 r_3 + (r_2)^2 > (r_2)^2 + (r_2)^2 + (r_2)^2 = 3(r_2)^2 > 3(r_1)^2$$

We have proved only the forward half of part 3).

To prove the other half, suppose one of the roots, call it  $r$ , is such that  $m > 3r^2$ . We need to establish there are two other roots, different from  $r$  and different from each other. Since  $r$  is a root we can write that  $mr + b = r^3$ , or that  $r^3 - mr - b = 0$ . Consider synthetic substitution where  $x = r$  and we substitute into the polynomial  $x^3 - mx - b$ .

$$\begin{array}{r|rrrr} & & & & r \\ & & & & \hline 1 & 0 & -m & -b & \\ & r & r^2 & r^3 - mr & \\ \hline 1 & r & r^2 - m & r^3 - mr - b & \end{array}$$

The last entry in the last row is 0. So we continue by considering the deflated polynomial in the last row  $x^2 + rx + (r^2 - m)$ . The discriminant of the deflated polynomial is:

$$r^2 - 4(r^2 - m) = -3r^2 + 4m > -3r^2 + 12r^2 = 9r^2$$

Now if  $r = 0$  then in the original polynomial we must have  $b = 0$  and this implies we have a line with positive slope  $m$  through the origin. In this special case, the three distinct roots are  $x = -\sqrt{m}$ , and  $x = 0$ , and  $x = \sqrt{m}$ .

On the other hand, if  $r \neq 0$  then the discriminant is positive and we can just apply the quadratic formula to determine the two roots other than  $r$ . The other two roots would be denoted and given by:

$$r_2 = \frac{-r - \sqrt{r^2 - 4(r^2 - m)}}{2} \text{ and } r_3 = \frac{-r + \sqrt{r^2 - 4(r^2 - m)}}{2}$$

Neither of these two roots is equal to  $r$  and neither root is equal to the other root. Thus we have exhibited three distinct roots whenever one root  $r$  is such that  $m > 3r^2$ .

Now that we have established five theorems that relate the properties and graphs of  $y = x^3$  and  $y = mx + b$  we can state why such objects are so fundamentally important for study. **Completely understanding these two simple objects and their relationship to each other is all that is needed to completely understand how to solve any general cubic polynomial equation.**

The most general form of a cubic equation is:  $ax^3 + bx^2 + cx + d = 0$

By dividing by  $a$  because it isn't zero, we can always assume the leading coefficient is 1 and thus reduce the general cubic to the more specific case that takes the form

$$x^3 + Bx^2 + Cx + D = 0$$

Now even this form can be simplified to eliminate the squared term by making a change of variable where we let  $x = z - \frac{B}{3}$ . After making this substitution into the last equation and simplifying we arrive at the cubic form:

$$z^3 + Pz + Q = 0.$$

If we can solve this equation for  $z$  we can solve the original general cubic equation for  $x$ . Of course the last two terms on the left,  $Pz + Q$  can be considered to make the equation of a line so the whole problem of a solving a cubic equation boils down to finding the points of intersection of the pure and simple cubic curve  $y = x^3$  with that of a general line we write in the form  $y = mx + b$ .

The astute reader who is interested in complex numbers should note that we have deliberately avoided talking about the nature of complex roots as they might relate to the general cubic equation. Of course, such roots occur in conjugate pairs and their study can be taken up after mastering the above five theorems. The deflated quadratic in the first part of **Theorem 5** is the source of all complex number roots of  $x^3 - mx - b = 0$ . Finding one real root  $r$  first allows us to determine the two complex roots in terms of  $r$  and  $m$ . In fact, the two equations at the end of the proof of part 3) in **Theorem 5** would yield the complex number solutions. In this case however, the discriminant of the reduced quadratic would be negative.

As a last comment, **Theorem 3** part 3) provides a means of constructing a tangent line to the curve  $y = x^3$ . The construction would begin by first choosing any  $y$ -intercept value  $b$  and marking the point  $(0, b)$  on the  $y$ -axis. Then reflect that point across the  $x$ -axis and construct the midpoint between the reflected  $y$ -axis point and the origin point. Finish the construction by drawing a horizontal line through the last  $y$ -axis midpoint. Where the horizontal line intersects the curve  $y = x^3$  is the point of tangency on that curve. The tangent line can now be drawn as the line through the first point  $(0, b)$  and the final intersection point. Try it, you'll like it. Were it not for the fact that this construction requires having the graph of  $y = x^3$  in the first place, this construction could almost be used as a ruler and compass construction for finding the cube root of a number.  $r_2$  is the cube root of  $-\frac{1}{2}b$ .

The above construction can be reversed. You could first choose the point on  $y = x^3$  at which you wish to find the tangent line. Then draw the horizontal line through that point and note where that horizontal line intersects the  $y$ -axis. Double the distance between that  $y$ -axis point and the origin, but moving away from the origin along the  $y$ -axis. Then reflect the doubled distance point across the  $x$ -axis. The final point on the  $y$ -axis can now be connected to the original point on the curve to make the tangent line.

After reading this paper you may wish to read another paper on the author's homepage web site that is titled *Solving Cubic Equations*. Just visit [http://homepage.smc.edu/kennedy\\_john](http://homepage.smc.edu/kennedy_john) and look under the link to **Downloadable Papers**.