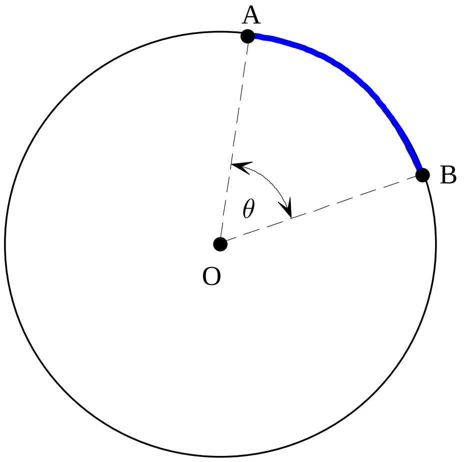
Basic Circle Facts

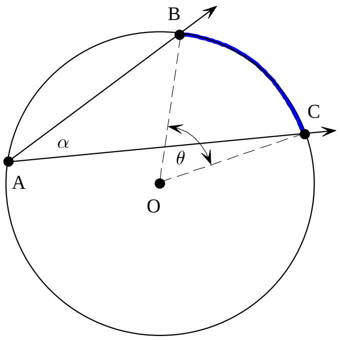
1. The degree measure of any arc on a circle is the same as the degree measure of the central angle determined by that arc.

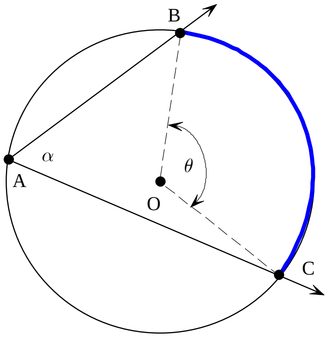
In the figure below, point O is the center of the circle. The above sentence is nothing more than a definition of how we measure an arc on a circle using angle measure. The central angle may be measured in either degrees or radians.



2. The measure of an angle inscribed in a circle is $\frac{1}{2}$ the measure of its intercepted arc.

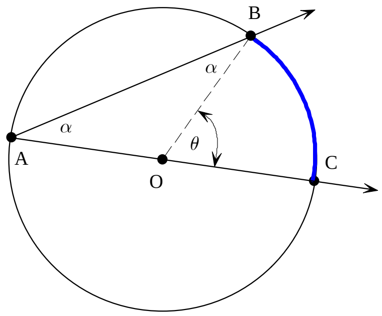
In the figure below, $m(\angle\alpha) = m(\angle BAC) = \frac{1}{2}m(\angle\theta) = \frac{1}{2}m\left(\widehat{BC}\right)$. In fact, there are three cases; the first case is where the circle center point O lies in the exterior of $\angle\alpha$; the second case is where the circle center point O lies on the line AC; the third case is where the circle center point O lies in the interior of $\angle\alpha$. The first and third cases are shown in the next two circle figures.



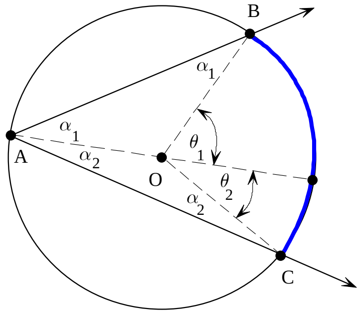


Proof of 2:

We will first prove the second case not shown in the above two circles. We begin by assuming the circle center point O lies on the line AC. In other words, we assume AC is a diameter of the circle. See the next circle in the figure below where the angle of interest is $\angle BAC$. Note that $\triangle AOB$ is isosceles because the two sides AO and OB are both radii. Thus the two base angles in $\triangle AOB$ can both be labeled as α. Finally, note that the angle θ is an exterior angle to $\triangle AOB$ at point O. This means we have $\theta = \alpha + \alpha = 2\alpha$. In other words, $\alpha = \frac{1}{2}\theta$.



For the third case where point O is in the interior of $\angle\theta$, we can split the inscribed angle θ into two angles, θ_1 and θ_2 as shown in the following figure. Then we apply the above argument to the two isosceles triangles, $\triangle AOB$ and $\triangle AOC$, that are formed on both sides of the diameter line that contains A and O.

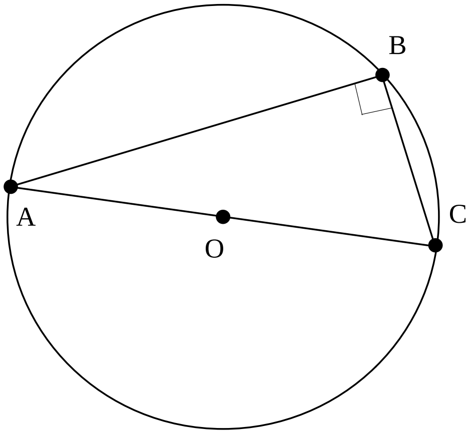


We won't give a proof of the first case for the first circle shown in property 2, but the secret is to draw AO and apply subtraction of angles and corresponding arcs.

3. An angle inscribed in a semi-circle is always a right angle.

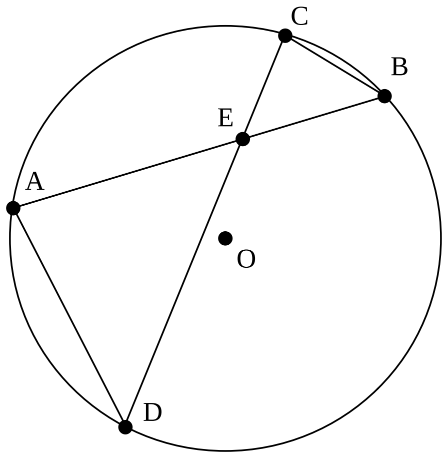
Proof of 3:

In the figure below, AC is a diameter of the circle and $\triangle ABC$ must be a right triangle with a right angle at point B. The reason is because $\angle B$ intercepts the arc AC that is half a circle. $m\left(\widehat{AC}\right) = 180^\circ$ and thus $m(\angle ABC) = \frac{1}{2}m\left(\widehat{AC}\right)=90^\circ$.



4. If two chords of a circle intersect at an interior point of that circle, then the two main triangles determined by those chords and their common point of intersection are similar triangles.

In the figure below, $\triangle AED \sim \triangle CEB$.



Proof of 4:

To establish the two triangles are similar we need only establish two equal angles in the two relevant triangles. Clearly, the pair of vertical angles are equal, $m(\angle AED) = m(\angle CEB)$ so we need only find one more pair of equal angles. That is simple however, because both $\angle EAD$ and $\angle ECB$ intercept the same large arc that is BD.

5. If two chords of a circle intersect at an interior point of that circle, then the product of the two lengths on one chord is the same as the product of the two lengths on the other chord.

For the above figure this statement says $(AE)(EB) = (DE)(EC)$.

Proof of 5:

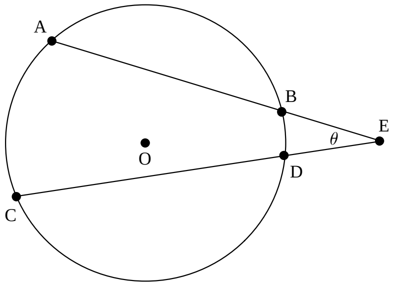
After establishing the two similar triangles, it is easy to setup the ratios of corresponding sides to derive the relevant equation.

$$\frac{AE}{DE} = \frac{EC}{EB} \quad \text{or} \quad (AE)(EB) = (DE)(EC).$$

6. If two secants of a circle intersect at a point exterior to the circle, then the angle at that point of intersection measures one half of the difference of the two intercepted arcs.

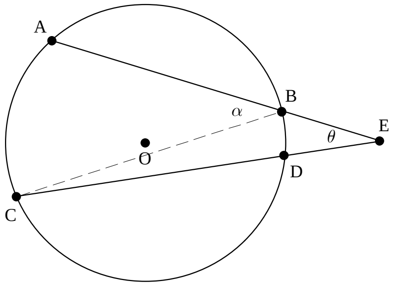
In the figure below the statement would be that

$$m(\angle\theta) = m(\angle AEC) = \tfrac{1}{2}m\left(\widehat{AC}\right) - \tfrac{1}{2}m\left(\widehat{BD}\right) = \tfrac{1}{2}\left\{m\left(\widehat{AC}\right) - m\left(\widehat{BD}\right)\right\}.$$



Proof of 6:

To establish the angle relationship we need only draw the chord CB and let $\angle\alpha = \angle ABC$. Then we can reason that α is exterior to $\triangle EBC$ at $B,$ so $m(\angle\alpha) = m(\angle BCD) + m(\angle\theta)$.

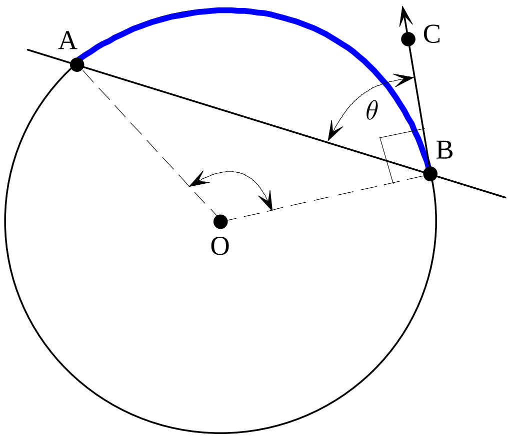


Now we simply solve for $m(\angle\theta)$.

$m(\angle\theta) = m(\angle\alpha) - m(\angle BCD) = m(\angle\alpha) - \tfrac{1}{2}m\left(\widehat{BD}\right) = \tfrac{1}{2}m\left(\widehat{AC}\right) - \tfrac{1}{2}m\left(\widehat{BD}\right) = \tfrac{1}{2}\left\{m\left(\widehat{AC}\right) - m\left(\widehat{BD}\right)\right\}.$

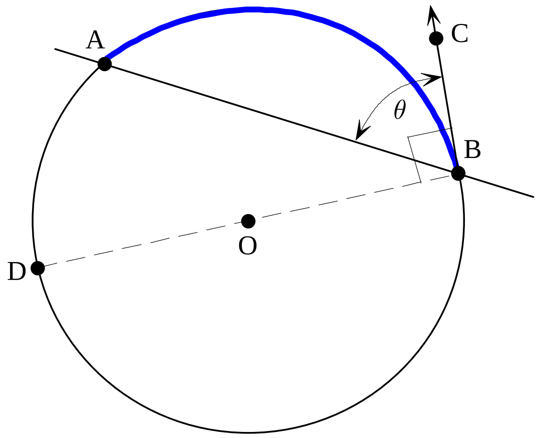
7. The measure of an angle formed by a secant and a tangent is equal to $\frac{1}{2}$ of the intercepted arc.

In the figure below, line BC is tangent to the circle at $B,$ and line AB is a secant line.
The statement is that $m(\angle\theta) = m(\angle ABC) = \frac{1}{2}m\left(\widehat{AB}\right)$.



Proof of 7:

Draw the diameter from B through the center point O and let point D be such that BD is a diameter.



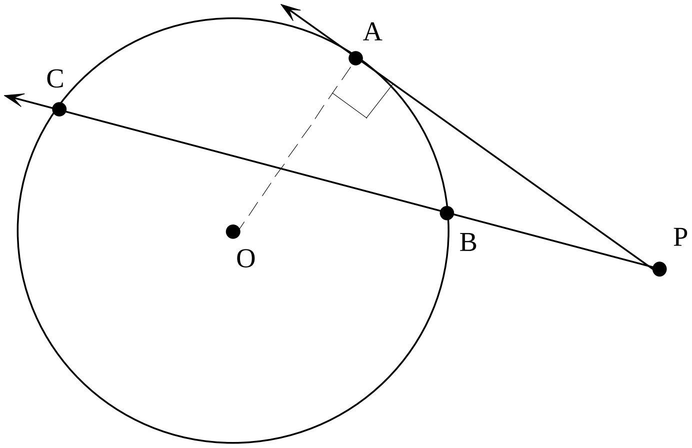
The derivation is simple because we know $\widehat{BAD}$ measures 180° and $\widehat{BAD} = \widehat{BA} + \widehat{AD}$. Also we know $m(\angle DBA) = \frac{1}{2}m\left(\widehat{AD}\right)$. Also, we have $m(\angle DBA) + m(\angle\theta) = 90^{\circ}$.
Now we just need to put all these equations together and solve for θ. We multiply this last equation by 2 on both sides and write:

$$\begin{gathered}
2 \cdot m(\angle DBA) + 2 \cdot m(\angle\theta) = 180^{\circ} = m\left(\widehat{AB}\right) + m\left(\widehat{AD}\right). \\
m\left(\widehat{AD}\right) + 2 \cdot m(\angle\theta) = m\left(\widehat{AB}\right) + m\left(\widehat{AD}\right) \\
2 \cdot m(\angle\theta) = m\left(\widehat{AB}\right) \\
m(\angle\theta) = \tfrac{1}{2}m\left(\widehat{AB}\right)
\end{gathered}$$

8. If a point P is outside a circle, and if both a tangent and a secant are drawn to the circle, then the square of the length of the tangent to that point is the same as the product of the two lengths from the point to the two intersection points determined by the secant.

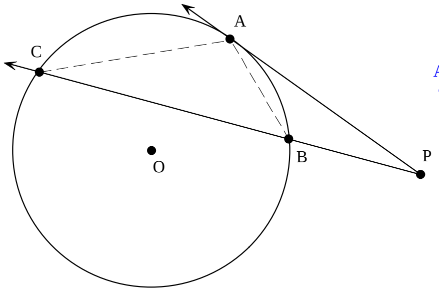
In the figure below we would write this statement as the proportion equation:

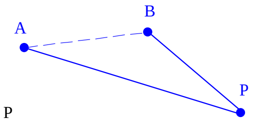
$$\frac{PA}{PC} = \frac{PB}{PA} \quad \text{or} \quad (PA)^2 = (PB)(PC)$$



Proof of 8:

Draw the segments AB and AC. Claim that $\triangle ACP \sim \triangle BAP$.



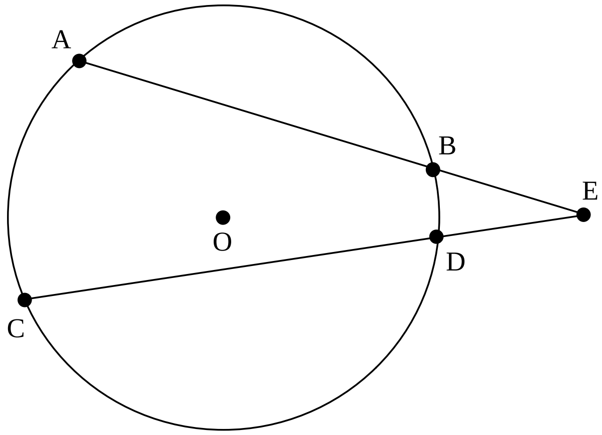


By the tangent/secant property we know $m(\angle BAP) = \frac{1}{2}m(\overset{\frown}{AB})$ and we know that $m(\angle ACP) = \frac{1}{2}m(\overset{\frown}{AB})$ so that $m(\angle ACP) = m(\angle BAP)$. Last, it is trivial that $m(\angle CPA) = m(\angle APB)$. Now by the triangle simliarity we can write:

$$\frac{PA}{PC} = \frac{PB}{PA} \quad \text{or} \quad (PA)^2 = (PB)(PC)$$

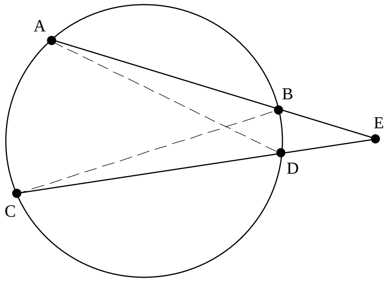
9. If two secants of a circle intersect at a point exterior to the circle, then the product of the length of each secant with the external length of that secant is the same for both secants.

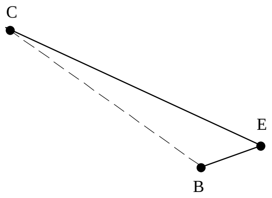
In the figure below this statement would say: $(AE)(BE) = (CE)(DE)$.



Proof of 9:

We draw the two segments BC and AD. Now we claim that $\triangle EAD \sim \triangle ECB$.





It is trivial that $m(\angle AED) = m(\angle BCE)$ because both of these angles measure $\frac{1}{2}$ of the same difference between the same two intercepted arcs. $m(\angle AED) = \frac{1}{2}\left\{m\left(\overset{\frown}{AC}\right) - m\left(\overset{\frown}{BD}\right)\right\}$ and $m(\angle BEC) = \frac{1}{2}\left\{m\left(\overset{\frown}{AC}\right) - m\left(\overset{\frown}{BD}\right)\right\}$ so these two angles measure the same.

Next, $m(\angle EAD) = \frac{1}{2}m\left(\overset{\frown}{BD}\right)$ and also $m(\angle BCE) = \frac{1}{2}m\left(\overset{\frown}{BD}\right)$. This gives us our second pair of equal angles. $m(\angle EAD) = m(\angle BCE)$. Now that the two triangles are similar we can write the following ratio:

$$\frac{AE}{CE} = \frac{DE}{BE} \quad \text{or} \quad (AE)(BE) = (CE)(DE)$$